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Localic sup-lattices and tropological systems ☆

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Abstract

The approach to process semantics using quantales and modules is topologized by considering tropological systems whose sets of states are replaced by locales and which satisfy a suitable stability axiom. A corresponding notion of localic sup-lattice (algebra for the lower powerlocale monad) is described, and it is shown that there are contravariant functors from sup-lattices to localic sup-lattices and, for each quantale Q, from left Q-modules to localic right Q-modules. A proof technique for third completeness due to Abramsky and Vickers is reset constructively, and an example of application to failures semantics is given.

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1. Introduction

In [2] and, subsequently, [11], different ideas of process semantics (in the computer science sense) are compared using the algebraic structures of quantales and modules over them.

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To a significant extent, that approach is already a topological one, at least in the localic sense (see, e.g. [6,15]). The quantales themselves are a generalization of frames—those complete lattices that in the localic account are the topologies—which yields a kind of noncommutative topology (see [8] for a survey), and moreover there arose numerous frames as topologies of spaces of processes.

However, there also remain places where the previous theory worked with untopologized sets. The aim of the present paper is to show how topologies may be incorporated, and to lay the ground for a more expressly topological development.

There is some interest in topologizing for its own sake. However, we have a deeper motivation: the failure to topologize is related to a failure to reason constructively, and we wish also to rectify that. As a general principle the relation is seen most clearly when the constructive discipline is the stringent *geometric* one, as used systematically in [17], giving a mathematics that is preserved by the inverse image functor of geometric morphisms. We want to use locales, for much of topology constructivizes more smoothly in localic form—as has been seen both in toposes [7] and in type theory [14]. However, constructing the *set* of points of a locale is not geometric: replacing the locale by the set of points (with its discrete topology) has not only changed the topology but has also made a constructive modification to the points that will show up when one tries to transfer to a different set theory along an inverse image functor. It follows that a constructive treatment should properly pay attention to the topology.

We remark that our notation and terminology for locales will always be such as to let them appear to be spaces. When we need to refer more concretely to the frame of opens of a locale X, we shall usually call it ΩX. Notation such as $f: X \to Y$ for locales will *always* mean a map (morphism) of locales, corresponding to a frame homomorphism $\Omega f: \Omega Y \to \Omega X$.

We write $\sqsubseteq$ for the order enrichment (the *specialization* order) of the category **Loc** of locales: if $f, g: X \to Y$, then $f \sqsubseteq g$ if for all $b \in \Omega Y$ we have $\Omega f(b) \leqslant \Omega g(b)$.

1.1. Sup-lattice duality

As a more concrete example, consider sup-lattice duality. This was given an intuitionistically constructive treatment in [7], and used heavily in [2]. If M is a sup-lattice—a complete join semilattice—then it also has all meets and so its opposite M^{op} is also a sup-lattice. Moreover, if $f: M \to N$ is a sup-lattice homomorphism—i.e., preserving all joins—then it has a right adjoint which preserves all meets, and hence corresponds to a sup-lattice homomorphism $f^{\mathrm{op}}: N^{\mathrm{op}} \to M^{\mathrm{op}}$. However, the application in [2] smuggled in classical reasoning by treating M^{op} as the set of sup-lattice homomorphisms from M to the two element chain 2. Then if M was viewed as a set of properties with a defined notion of disjunction (the joins), M^{op} could be viewed as the set of models of those properties. But there is a problem here constructively. An element of M^{op} is a function from 1 to M^{op}, and hence corresponds to a sup-lattice homomorphism from $\Omega = \mathscr{P}1$ to M^{op}, since [7] a powerset $\mathscr{P}X$ is the free sup-lattice over X. Dualizing, this corresponds to a homomorphism from M to Ω^{op}, not to Ω. Since Ω^{op} is not the lattice of truth values, it is wrong to view such a homomorphism as a model of the properties in M. We repair this by replacing the [7] dual, a *set*, by a *locale* $\hat{M}$ for which M provides a base of opens and the points of $\hat{M}$ are the sup-lattice homomorphisms from M to Ω. In Section 3 we shall expound a corresponding theory of "localic sup-lattices" which will underlie our topologized approach to process semantics.

1.2. Tropological systems

In those earlier papers [2,11] the fundamental model of "process" is taken to be the *labelled transition system* or *LTS*, that is to say (with respect to a set *Act* of "actions") a set P (of "states") equipped with an *Act*-indexed family of binary relations $\xrightarrow{\alpha} \subseteq P\times P$ ($\alpha\in Act$). In the concurrency literature various equivalences have been defined on states of LTSs (see, e.g. [4,5]), and the aim has been to characterize a significant number of such equivalences in terms of the kind of observations that can be made on LTSs: more observations will lead to a finer equivalence.

The actions themselves can be understood as observations ("see α happening"), but their dynamic nature means that one has to be explicit about the order and multiplicity of the observations, and they are taken algebraically as being elements of a (unital) *quantale*, a sup-lattice equipped with monoid structure for which multiplication distributes over arbitrary joins. Other observations may also be available in particular computing contexts. In particular, *acceptances* $\alpha^{\checkmark}$ indicate that α is possible without actually doing it (for instance, one might see it on a menu), and *refusals* $\alpha^{\times}$ indicate that α is impossible (for instance, one might see it "greyed out" on a menu). Observing an acceptance or refusal does not (unlike observing an action) change the state of the system.

The natural order on the quantale refers to both "before" and "after" states for an observation: $a\leqslant b$ means that whenever p can change to q with a observed, then p can also change to q with b observed. But it is also necessary to treat the observations as determining properties of the "before" state: if p changes to something with a observed then a "is possible" for p, and we also need an order $a\leqslant' b$ meaning that for all states p, if a is possible for p then so is b. This is conveniently handled by using a *left module* over the quantale in a sense exactly analogous to that of modules over rings, and leads to the *tropological systems* of [11], systems combining a process set P with a quantale Q and a left Q-module L (generalizing Q' in [2]). This involves a quantale homomorphism from Q to the relational quantale $\mathscr{P}(P\times P)$ (in which multiplication is relational composition) and a module homomorphism Π from L to the left $\mathscr{P}(P\times P)$-module $\mathscr{P}P$. We write $p\models x$ if $p\in\Pi(x)$, in which case we say that p satisfies the property corresponding to x.

In Section 2 we provide some background on tropological systems, presenting them in a way that fits the purposes of this paper.

1.3. Completeness

In [2] three notions of completeness (two of them being completeness of the relations used to present quantales and left modules) are defined and proved in several process theoretic examples. From this point of view, LTSs are “models” in a logical sense, for the notions of completeness are defined by reference to them.

In any LTS P, the interpretations of acceptances and refusals are determined by the transition relations $\xrightarrow{\alpha}$. If Q and L are then chosen to be generated by the particular combination of operators available in some given computing context then we find, as *unique extension theorems*, that P can be made into a tropological system with Q and L in a unique way [11,9]. This then leads to a preorder on the elements of P: $p \lesssim q$ if, for every $x \in L$, if $p \models x$ then $q \models x$. The “first completeness” results in [2,11] were to show that this preorder coincided with some process preorder already known.

In addition, further “second completeness” and “third completeness” results concerned the way that Q and L were presented by generators and relations, specifically, that the relations were complete with respect to LTSs. Second completeness says that for all $a, b \in Q$, if for every LTS P and for every $p, q \in P$ we have $p \xrightarrow{a} q$ implies $p \xrightarrow{b} q$, then $a \leqslant b$. In other words, if a is *semantically* less than b—with respect to all the LTSs as models—, then it is *syntactically* less than b—with respect to the presenting relations that govern the order in Q itself. Third completeness is similar for L. It says that for all $x, y \in L$, if for every LTS P and for every $p \in P$ we have $p \models x$ implies $p \models y$, then $x \leqslant y$.

In this paper we shall largely concern ourselves with third completeness, so let us outline the classical argument that appeared in [2]. Given Q and L, let L^{op} be the sup-lattice dual to L—as a sup-lattice it is L with the opposite order, and is a right Q-module. Certain elements of L^{op} were then defined to be “pointlike”, and it was shown that (i) they form an LTS, a “master transition system” *Proc*, and (ii) (using the axiom of choice) every element of L^{op} is a join of pointlikes. This then is classically enough to show third completeness, for if $x \nleqslant y$ in L then $y \nleqslant x$ in L^{op}, there is some pointlike $p \leqslant y$ with $p \nleqslant x$, and that shows that $p \models x$ but $p \not\models y$.

1.4. Localic transition systems

It seems that the above argument is inextricably classical, but in Section 4.4 we shall give a constructive development that relies in part on using localic transition systems: the set P is to be replaced by a locale P. This idea goes back to [1] and such systems appeared in [2,11] and were defined implicitly in the “S-frames” and “RS-frames”. The transition structure is defined by sup-lattice endomorphisms $\langle\alpha\rangle$ on ΩP, in other words continuous locale maps from P to the lower powerlocale $\mathrm{P_L}P$—conceptually, a point of P maps to the sublocale of those points to which it can undergo a transition under α. However, whereas in a classical LTS we can always find the complements needed to interpret refusals, this is not possible for the locales in general. We therefore get a range of different kinds of localic transition systems. The “S-systems” (S for simulation) simply have the sup-lattice endomorphisms $\langle\alpha\rangle$, while in an “RS-system” (RS for ready simulation) the frame elements $\langle\alpha\rangle\top$ are required to have Boolean complements. In addition, “B-systems” (for bisimulation) have been studied in [11]. We treat the S-, RS- and B-systems as a localic splitting of the classical idea of LTS and regard them as the models with respect to which we define completeness. In Section 5.1 we provide a brief description of these systems. All kinds of systems are appropriate to the T (trace), A (acceptance), AT (acceptance trace) and S (simulation) semantics, whereas for F (failure), FT (failure trace), RT (ready trace) and RS (ready simulation) semantics, RS-or B-systems are needed, and for B (bisimulation) semantics, B-systems are needed. We leave the detailed proof of this fact to a subsequent paper, but provide a brief explanation in Section 5.2.

An interesting consequence of the localic approach is that we now have *final* S-, RS- and B-systems Tr_{S}, Tr_{RS}, and Tr_{B}. We call these *tree locales*, since, essentially, their points are synchronization trees modulo simulation, ready simulation and bisimulation. Another such locale is constructed in [1], covering systems with divergence. As final systems, they provide semantic domains for transition systems of the appropriate kinds. By definition we have maps $Tr_{\mathrm{B}} \to Tr_{\mathrm{RS}} \to Tr_{\mathrm{S}}$, and we conjecture that they are localic surjections. If this holds, then where there is a choice of which kind of system to use, it makes no difference from the point of view of third completeness.

1.5. Localic tropological systems

In order to proceed we also need to redefine tropological systems in a localic setting. In Section 4 we present a corresponding notion of localic tropological system, in which the set of states is replaced by a locale. In order to guarantee that the states of such systems behave appropriately, we impose a new stability axiom (which is shown to be trivial in the classical theory), of which we give a detailed study in Section 4.2. The localic setting provides us with algebraic tools otherwise unavailable, such as the possibility of presenting systems by generators and relations. In particular this automatically yields notions of final semantics, which we discuss in Section 4.3, and as a consequence leads to more convenient definitions of second and third completeness, described in Section 4.4, whereby it is only required that certain maps be 1-1. There are also unique extension theorems, for the T, A, F, R, AT, FT, RT, S, RS, and B semantics, which relate localic transition systems and localic tropological systems, and depend crucially on the stability axiom. We explain the main ideas behind this in Section 5.2, where we also discuss the significance of the theorems for completeness, but defer the detailed proofs to a subsequent paper.

The strategy for proving third completeness now requires that we convert L^{op} and *Proc* to locales. L^{op} will be replaced by the sup-lattice dual $\hat{L}$ already mentioned. Then we define *Proc* to be a particular sublocale of $\hat{L}$ and show it to be a localic transition system of the appropriate kind. As already stated, third completeness means that certain functions are 1-1, and this will correspond to certain maps between locales being surjections. One of these in particular will correspond naturally to the classical lemma that elements of L^{op} are joins of pointlikes, but instead of using the axiom of choice to prove this (part of which is using choice to prove the sufficiency of points in a locale) we shall define maps by a constructive manipulation of presentations of the locales. This is described in Section 4.4, and in Section 5.3 we apply this technique to the example of failures semantics F.

2. Background

In this section we introduce technical preliminaries, terminology and notation, along with an overview of tropological systems meant to present them in a way which is appropriate for the applications in the later sections.

2.1. Preliminaries

The category **SL** [7] has the complete lattices—*sup-lattices*—as objects and the maps that preserve arbitrary joins as morphisms—*sup-lattice homomorphisms*. A homomorphism $f : L \to M$ is *strong* if it preserves the top: $f(\top_L) = \top_M$. We denote the minimum element of a sup-lattice by 0.

The *tensor product* of sup-lattices $L \otimes M$ is characterized by the property that homomorphisms from it to any sup-lattice N are equivalent to "bilinear" functions from $L \times M$ to N, i.e., those that preserve joins in each component separately. This makes **SL** a monoidal category [7], and the monoids in it are the *unital quantales*, i.e., sup-lattices equipped with an associative multiplication $\cdot$ (with a unit—usually denoted by 1) that distributes over all joins in both variables. A *unital right module* over a quantale Q, or *(unital) right Q-module*, is a unital right action (usually also denoted by $\cdot$) over the monoid Q in **SL**, and similarly for left modules. *Homomorphisms* of unital quantales are maps that preserve the monoid structure and all the joins, and *homomorphisms* of modules, for each Q fixed, are maps that commute with the action and preserve all the joins (the definitions are therefore entirely analogous to those for rings and ring modules, except that the underlying abelian group structure has been replaced by the sup-lattice structure). We shall also work extensively with strong homomorphisms between unital left Q-modules, with respect to which the module $Q \cdot \top = \{a \cdot \top \mid a \in Q\}$, whose action is multiplication on the left, is an initial object; that is, given any other unital left Q-module M there is a unique strong homomorphism $Q \cdot \top_Q \to M$ of left Q-modules [11, Proposition 3.6(8)]. Further details about quantales and modules can be found in [2,11,12,13].

We denote the category of unital quantales by **Qu**. Other categories that we shall mention in this paper are: **DL** (bounded distributive lattices and their homomorphisms); **Fr** (frames and their homomorphisms); $\vee$-**sL** (join semilattices, with bottom, and their homomorphisms); $\wedge$-**sL** (meet semilattices, with top, and their homomorphisms). All these categories are algebraic, and for a presentation by generators and relations we usually write $C\langle G \mid R\rangle$, where C is the name of the corresponding category, italicized, G is the set of generators, and R is the set of defining relations of the presentation; if $R = \emptyset$ we write only $C\langle G\rangle$. For instance, $Qu\langle G \mid R\rangle$ is the unital quantale generated by G with relations in R. Often, too, G should not be taken to be just a set, but rather an object of another category D, whose structure should be preserved in the presentation. Instead of adding this restriction to the sets of relations we write instead $C\langle G(\text{qua } D) \mid R\rangle$ to indicate that the structure of G as an object of D should be preserved. For instance, if L is a sup-lattice we write $Fr\langle L(\text{qua } \mathbf{SL})\rangle$, or $Fr\langle L(\text{qua sup-lattice})\rangle$, for the frame freely generated by L as a sup-lattice (i.e., whose injection of generators is a universal sup-lattice homomorphism).

Let S be a poset, and let C be a *precoverage* on S, by which we mean a function assigning a set of subsets of $\downarrow(x)$ to each $x \in S$. The sup-lattice

$$SL\,\langle S \text{ (qua poset)} \,|\, x = \bigvee U[U \in C(x)]\rangle$$

can be concretely described [2] as the set of C-ideals of S, where by a C-*ideal* we mean a lower closed subset $J \subseteq S$ such that $x \in J$ whenever $U \subseteq J$ and $U \in C(x)$.

An important technical tool is the following version of Johnstone's [6] coverage theorem, where by a *coverage* C on a meet semilattice S we mean a precoverage on S such that whenever $U \in C(x)$ then

$$\{y \wedge u \,|\, u \in U\} \in C(y \wedge x) \text{ ("meet-stability").}$$

Theorem 2.1. *Let S be a meet semilattice, and let C be a coverage on S. Then,*

$$Fr\,\langle\, S \text{ (qua } \wedge\text{-}\mathbf{sL}) \,|\, x = \bigvee U[U \in C(x)]\rangle$$

is order-isomorphic to

$$SL\,\langle S \text{ (qua poset)} \,|\, x = \bigvee U[U \in C(x)]\rangle\,.$$

Proof. See [2]. $\square$

2.2. Tropological systems

Now we recall the notion of system of [2,11], where unital quantales are algebras of *finite observations* and unital left quantale modules are algebras of *capabilities* (also sometimes called "finitely observable properties"). We refer the reader to [2] or [11] for further motivation regarding these ideas, or to the short survey in [9].

The underlying notion of system in [2,11] is that of tropological system [11], which can be presented as being a structure (P, Q, L, Π) consisting of

- A set P (of states);
- A unital quantale Q (of finite observations);
- A unital left Q-module L (of finitely observable properties);
- A unital left action of Q on $\mathscr{P}P$;
- A strong homomorphism of left Q-modules $\Pi : L \to \mathscr{P}P$.

The left Q-module structure on $\mathscr{P}P$ defines, for each observation $a \in Q$, a binary relation $\xrightarrow{a} \subseteq P \times P$, the *transition relation* of a, by

$$p \xrightarrow{a} q \quad \Leftrightarrow \quad p \in a \cdot \{q\},$$

where $p \xrightarrow{a} q$ can be read as saying that if the system is at state p then (i) a can be observed and (ii) after it has been observed the resulting state may be q. Conversely, the left action can be recovered from the transition relation:

$$a \cdot X = \{p \in P \,|\, \exists q \in X.\, p \xrightarrow{a} q\}.$$

The homomorphism Π gives us, for each property $\varphi \in L$, the set of states where φ holds. We also write $p \models \varphi$ when $p \in \Pi(\varphi)$, and call the binary relation $\models\ \subseteq P \times L$

thus defined the *satisfaction relation* of the system. When $p \models \varphi$ we say that p *satisfies* φ.

We call a structure (P, Q, L, Π) as above, but where Π is not required to be strong, a *pre-tropological system*.

We shall take the view that Q and L provide information about single "observational runs" on processes. The elements of L denote properties of the starting state of the run, as observably known at the end of the run, and Q acts on L by prefixing observation steps from Q: $a \cdot \varphi$ means "after a, φ is possible" and to observe it from the starting state p you first observe a happening, moving to some state q, and then observe the property φ starting from q. (We shall later—Section 4.1—argue that repeated observational runs correspond to logical conjunction.) The following properties of transition and satisfaction relations bring out these intuitions more clearly, and hold of any tropological system:

- $p \xrightarrow{1} q$ if and only if $p = q$,
- $p \xrightarrow{a \cdot b} q$ if and only if $p \xrightarrow{a} r \xrightarrow{b} q$ for some $r \in P$,
- $p \xrightarrow{\bigvee X} q$ if and only if $p \xrightarrow{a} q$ for some $a \in X$,
- $p \models \top$,
- $p \models a \cdot \varphi$ if and only if $p \xrightarrow{a} q$ and $q \models \varphi$ for some $q \in P$,
- $p \models \bigvee Y$ if and only if $p \models \varphi$ for some $\varphi \in Y$.

In fact, the structure (P, Q, L, Π) is a tropological system if and only if the above six conditions hold [11], and the strength of Π is equivalent to the fourth condition.

Any sup-lattice homomorphism $f : L \to M$ has a right adjoint $f_* : M \to L$ that preserves meets and thus defines a sup-lattice homomorphism $f^{\mathrm{op}} : M^{\mathrm{op}} \to L^{\mathrm{op}}$; moreover, the passage $f \mapsto f^{\mathrm{op}}$ preserves joins. It follows that if L is a left Q-module then L^{op} is a right Q-module (a similar argument will appear in Corollary 3.5), and from the classical fact that set complementation makes $\mathscr{P}P$ self-dual it follows that if (P, Q, L, Π) is a pre-tropological system then $\mathscr{P}P$ is also a right Q-module. Its action is referred to as the *dynamics*, and it is defined explicitly as follows.

$$\begin{aligned} X \cdot a \quad &= \quad \{q \in P \mid \exists p \in X. p \xrightarrow{a} q\}, \\ p \xrightarrow{a} q \quad &\Leftrightarrow \quad q \in \{p\} \cdot a. \end{aligned}$$

Moreover, using again set complementation we obtain a homomorphism of right Q-modules

$$\mathscr{P}P \xrightarrow{\cong} (\mathscr{P}P)^{\mathrm{op}} \xrightarrow{\Pi^{\mathrm{op}}} L^{\mathrm{op}},$$

which can be identified with a map $K : P \to L^{\mathrm{op}}$ because $\mathscr{P}P$ is freely generated by P as a sup-lattice. Explicitly, K is given by

$$K(p) = \bigvee \{\varphi \in L \mid p \not\models \varphi\}$$

(the join is calculated in L, not in L^{op}). Using again classical reasoning, L^{op} is order isomorphic to the sup-lattice $\check{L}$ whose elements are the complements of the principal ideals of L, ordered under inclusion, and if we substitute $\check{L}$ for L^{op} in the definition of K we obtain

$$K(p) = \{\varphi \in L \mid p \models \varphi\}.$$

Hence, $K(p)$ contains the information of which properties $\varphi \in L$ are satisfied by $p \in P$—the "capabilities" of p—and hence expresses a semantics by capabilities.

$\check{L}$ (and indirectly L^{op}) thus contains meanings of states ("processes"), but also the meanings of sets of states, calculated as joins. In particular, the bottom element $\emptyset \in \check{L}$ is the meaning of the empty set of states. We normally reinforce this idea by requiring the states $p \in P$ to have $K(p) \neq \emptyset$, which is equivalent to Π being strong.

Hence, another way of presenting tropological systems consists of the following data:

- A set P;
- A unital quantale Q;
- A unital left Q-module L;
- A Q-indexed family of maps $P \to \mathscr{P}P$ that jointly define a right Q-module structure on $\mathscr{P}P$ (the dynamics);
- A map $K : P \to \check{L}$ that extends to a right Q-module homomorphism $\mathscr{P}P \to \check{L}$ and factors via $\check{L} \backslash \emptyset$.

A pre-tropological system is the same, except that K is not required to factor via $\check{L} \backslash \emptyset$.

The purpose of the present section has been to give some background on tropological systems, and we conclude with a brief remark about some generalizations, which however will not be further addressed in the present paper. For instance, it is possible to allow the module structures of both L and $\mathscr{P}P$ to be pre-unital [9,10] (i.e., satisfying only $1 \cdot \varphi \geqslant \varphi$ and $X \cdot 1 \supseteq X$, respectively) in order to deal with systems that have hidden unobservable behaviour, which is equivalent to replacing the first of the three properties of the transition relation stated above by the weaker

$$p \xrightarrow{1} p \quad \text{for all } p \in P.$$

Other generalizations can be obtained by replacing $\mathscr{P}P$ by other kinds of lattices, such as lattices of projections on a Hilbert space for systems with quantum behaviour (see [9] for a preliminary account of this), which may also bring tropological systems closer to the study of points of quantales [10].

3. Localic sup-lattices

A poset X is a sup-lattice if and only if the function $\downarrow : X \to \mathscr{L}X$ has a left adjoint, $\mathscr{L}X$ being the set of lower closed subsets of X. The corresponding localic construction is the lower powerlocale $\mathrm{P_L}$, defined by

$$\Omega \mathrm{P_L} X = Fr\langle \Omega X (\text{qua } \mathbf{SL})\rangle.$$

If a is an open for X then we write $\Diamond a$ for the corresponding subbasic open of $\mathrm{P_L}X$; hence, "qua $\mathbf{SL}$" means then that $\Diamond$ preserves all joins. The points of $\mathrm{P_L}X$ can be identified with the "weakly closed sublocales of X with open domain" [3,16]. For present purposes, however, we do not need this precise characterization and it is not too misleading to think of the points in the classically equivalent way as the closed sublocales.

$\mathrm{P_L}$ is the functor part of a KZ-monad on **Loc**. Following [18], and in imitation of the poset notation, we shall write $\downarrow$ and $\sqcup$ for the unit and multiplication of the monad. Then a locale X is a $\mathrm{P_L}$-algebra (in a unique way) if and only if $\downarrow_X$ has a left adjoint, which we shall write as $\sqcup$. We call $\mathrm{P_L}$-algebras *localic sup-lattices*, and write **LocSL** for the category of them.

Proposition 3.1. **LocSL** *is sup-lattice enriched, the order being the specialization order* $\sqsubseteq$.

Proof. First, note that the Kleisli category for $\mathrm{P_L}$ is sup-lattice enriched: for it is dual to the full subcategory of **SL** whose objects are frames. The ordering is $\sqsubseteq$, for if $f, g : X \to \mathrm{P_L} Y$ then

$$\begin{aligned} f \sqsubseteq g \;&\Leftrightarrow\; \forall c \in \Omega \mathrm{P_L} Y. \Omega f(c) \leqslant \Omega g(c) \\ &\Leftrightarrow\; \forall b \in \Omega Y. \Omega f(\Diamond b) \leqslant \Omega g(\Diamond b). \end{aligned}$$

Now suppose that Y is a localic sup-lattice. We show that for any locale X, the homset $\mathbf{Loc}(X, Y)$ is a sup-lattice with respect to $\sqsubseteq$. For let $\varphi_i : X \to Y$ $(i \in I)$, let $\varphi' : X \to \mathrm{P_L} Y$ be the join of the Kleisli morphisms $\varphi_i; \downarrow$ and let $\varphi = \varphi'; \sqcup : X \to Y$. We have $\varphi_i = \varphi_i; \downarrow; \sqcup \sqsubseteq \varphi'; \sqcup = \varphi$, and if every $\varphi_i \sqsubseteq \psi$ then $\varphi_i; \downarrow \sqsubseteq \psi; \downarrow$, so $\varphi' \sqsubseteq \psi; \downarrow$ and by the adjunction $\sqcup \dashv \downarrow$, $\varphi \sqsubseteq \psi$. Hence φ is the join in $\mathbf{Loc}(X, Y)$ of the φ_is.

If $\chi : W \to X$, then for any map $\alpha : X \to \mathrm{P_L} Y$ the composite $\chi; \alpha$ is equal to the Kleisli composite of $\chi; \downarrow$ with α, and it follows that $\chi; \varphi'$ is the join of the maps $\chi; \varphi_i; \downarrow$ and hence that $\chi; \varphi$ is the join of the maps $\chi; \varphi_i$: precomposition by arbitrary maps distributes over the joins. For postcomposition, by $\omega : Y \to Z$, we must take Z also to be a localic sup-lattice and ω to be a homomorphism. The Kleisli compositions of $\varphi_i; \downarrow$ and of φ' with $\omega; \downarrow$ are $\varphi_i; \omega; \downarrow$ and $\varphi'; \mathrm{P_L}\omega$, respectively, and we deduce the join of the maps $\varphi_i; \omega$ is $\varphi'; \mathrm{P_L}\omega; \sqcup = \varphi'; \sqcup; \omega = \varphi; \omega$.

Finally, we must show that if X too is a localic sup-lattice and every φ_i is a homomorphism then so is φ, i.e., $\sqcup; \varphi = \mathrm{P_L}\varphi; \sqcup$. The $\sqsupseteq$ direction holds if and only if

$$\mathrm{P_L}\alpha \sqsubseteq \sqcup; \alpha; \downarrow = \sqcup; \downarrow; \mathrm{P_L}\alpha$$

which is obvious. Hence φ is a homomorphism if and only if $\sqcup; \varphi \sqsubseteq \mathrm{P_L}\varphi; \sqcup$. But $\sqcup; \varphi$ is the join of the maps $\sqcup; \varphi_i$, and $\sqcup; \varphi_i = \mathrm{P_L}\varphi_i; \sqcup \sqsubseteq \mathrm{P_L}\varphi; \sqcup$.

This proves the result: every homset of **LocSL** is a sup-lattice with respect to $\sqsubseteq$, and composition distributes over joins on both sides. $\square$

If Q is a quantale, we can also define the notion of localic Q-module as a localic sup-lattice L equipped with a quantale homomorphism from Q to $\mathbf{LocSL}(L, L)$. To distinguish right from left, we must be careful to distinguish the two categorical orders of composition: the *diagrammatic* order $f;g$ is the one that requires the target of f to be the source of g, while the *applicative* order is $g \circ f = f;g$.

Definition 3.2. Let Q be a quantale. A localic sup-lattice L is a *right localic Q-module* when it is equipped with a quantale homomorphism

$$\rho : Q \to (\mathbf{LocSL}(L,L), ;)$$

If $a \in Q$ and $x : 1 \to L$ is a point of L (or, more generally, if $x : X \to L$ is a generalized point at stage of definition X), then we write $x \cdot a$ for $x; \rho(a)$.

Similarly, L is a *left* localic Q-module when it is equipped with a quantale homomorphism

$$\lambda : Q \to (\mathbf{LocSL}(L,L), \circ)$$

and then we write $a \cdot x$ for $x; \lambda(a) = \lambda(a) \circ x$.

Definition 3.3. Let M be a sup-lattice. Then its *localic sup-lattice dual*, $\hat{M}$, is defined by

$$\Omega\hat{M} = Fr\langle M(\text{qua } \mathbf{SL})\rangle.$$

We shall write $\Diamond x$ for $x \in M$ considered as a generator of $\Omega\hat{M}$. This extends the notation used for subbasic opens of the lower powerlocale, for which we have $\mathrm{P_L}X = \widehat{\Omega X}$.

Notice that the points of $\hat{M}$ are the sup-lattice homomorphisms from M to Ω. *Classically*, $\Omega \cong \Omega^{\mathrm{op}}$ and so those points are equivalent to sup-lattice homomorphisms from Ω to M^{op}, i.e., to functions from 1 to M^{op} (since for any set X we have that $\mathscr{P}X$ is the free sup-lattice over X), i.e., elements of M^{op}. (Compare with the lower powerlocale, where classically the points of $\mathrm{P_L}X$ are the elements of ΩX^{op}, i.e., the closed sublocales of X. But constructively the points are less simply described [3,16].) Hence this localic sup-lattice dual is a localic analogue of the ordinary sup-lattice dual, and in fact it seems that the purposes in [2] for which the sup-lattice dual was used are better served by the localic sup-lattice dual. We do not claim that it provides a duality between **SL** and **LocSL**.

Proposition 3.4. *The construction* ˆ *provides a contravariant, sup-lattice enriched functor from* **SL** *to* **LocSL**.

Proof. First we show that if M is a sup-lattice then $\hat{M}$ is a localic sup-lattice. Define $\sqcup : \mathrm{P_L}\hat{M} \to \hat{M}$ by $\Omega \sqcup (\Diamond x) = \Diamond\Diamond x$. We have $\Omega\downarrow \circ\, \Omega\sqcup(\Diamond x) = \Diamond x$, and so $\downarrow; \sqcup = \mathrm{Id}_{\hat{M}}$. A subbase of opens for $\mathrm{P_L}\hat{M}$ is provided by the opens $\Diamond(\Diamond x_1 \wedge \cdots \wedge \Diamond x_n)$ and

$$\begin{aligned}\Omega \sqcup \circ \Omega \downarrow (\Diamond(\Diamond x_1 \wedge \cdots \wedge \Diamond x_n)) &= \Omega \sqcup (\Diamond x_1 \wedge \cdots \wedge \Diamond x_n) \\ &= \Diamond\Diamond x_1 \wedge \cdots \wedge \Diamond\Diamond x_n \geqslant \Diamond(\Diamond x_1 \wedge \cdots \wedge \Diamond x_n)\end{aligned}$$

Hence $\sqcup; \downarrow \sqsupseteq \mathrm{Id}_{\mathrm{P_L}\hat{M}}$ and it follows that $\sqcup$ is left adjoint to $\downarrow$.

If $f : M \to N$ is a sup-lattice homomorphism, then we define $\hat{f} : \hat{N} \to \hat{M}$ by $\Omega\hat{f}(\Diamond x) = \Diamond f(x)$. This is a $\mathrm{P_L}$-homomorphism, for

$$\begin{aligned}\Omega \sqcup \circ \Omega\hat{f}(\Diamond x) &= \Omega \sqcup (\Diamond f(x)) = \Diamond\Diamond f(x)\\ &= \Diamond\Omega\hat{f}(\Diamond x) = \Omega\mathrm{P_L}\hat{f}(\Diamond\Diamond x) = \Omega\mathrm{P_L}\hat{f} \circ \Omega \sqcup (\Diamond x)\end{aligned}$$

and it follows that $\sqcup; \hat{f} = \mathrm{P_L}\hat{f}; \sqcup$. The construction is clearly functorial.

To show that $\hat{}$ is sup-lattice enriched, let $\varphi_i : M \to N$ $(i \in I)$ be sup-lattice homomorphisms. The proof of Proposition 3.1 showed that the join $\bigvee_i \widehat{\varphi}_i$ in $\mathbf{LocSL}(\hat{N}, \hat{M})$ is also a join in $\mathbf{Loc}(\hat{N}, \hat{M})$, which is order isomorphic to $\mathbf{SL}(M, \Omega\hat{N})$, corresponding to the join of the sup-lattice homomorphisms $\Diamond; \Omega\widehat{\varphi}_i = \varphi_i; \Diamond$. That join is $(\bigvee_i \varphi_i); \Diamond$, corresponding to $\widehat{\bigvee_i \varphi_i}$. □

Corollary 3.5. *If Q is a quantale and M is a left Q-module, then $\hat{M}$ is a localic right Q-module.*

Proof. The left Q-module structure on M is given by a quantale homomorphism from Q to the endomorphism quantale $\mathbf{SL}(M, M)$ (with multiplication being composition in applicative order: $f \cdot g = f \circ g$). The functor $\hat{}$ maps this contravariantly to $\mathbf{LocSL}(\hat{M}, \hat{M})$, preserving sups, and thus makes $\hat{M}$ a right Q-module. □

Proposition 3.6. *Let M be a sup-lattice, X a locale, $f : X \to \hat{M}$ a map and $\bar{f} : \mathrm{P_L}X \to \hat{M}$ its lifting to a localic sup-lattice homomorphism. Then $\bar{f} = \widehat{\Omega f \circ \Diamond}$. (Note that we are using $\mathrm{P_L}X = \widehat{\Omega X}$.)*

Proof. We have $\bar{f} = \mathrm{P_L}f; \bigsqcup$, so we can calculate

$$\Omega\bar{f}(\Diamond x) = \Omega\mathrm{P_L}f(\Diamond\Diamond x) = \Diamond\Omega f(\Diamond x). \qquad \square$$

One might expect the injection of generators, $M \to \Omega\hat{M}$, to be 1-1. It seems to be an open question whether this is indeed the case in general, but we shall prove some special cases. Some of these will be used later.

Proposition 3.7. *Let M be a sup-lattice, and $\Diamond : M \to \Omega\hat{M}$ the injection of generators.*

1. *If M is a frame then $\Diamond$ is* 1-1.
2. *Classically, $\Diamond$ is* 1-1 *for every M.*
3. *If M is an algebraic lattice then $\Diamond$ is* 1-1. *(An "algebraic lattice" is the ideal completion of a join semilattice.)*

Proof. (1) The universal property of $\Omega\hat{M}$ gives a frame homomorphism $\Omega\hat{M} \to M$ that splits $\Diamond$.

(2) Classically Ω is 2, so suppose $\Diamond x \leqslant \Diamond y$ in $\Omega\hat{M}$ and define a suplattice homomorphism $f : M \to 2$ such that $f(z) = 0$ if and only if $z \leqslant y$. Then f extends to a frame homomorphism $f' : \Omega\hat{M} \to 2$, and $f(x) = f'(\Diamond x) \leqslant f'(\Diamond y) = f(y) = 0$, so $x \leqslant y$.

(3) Note that our dual $\hat{M}$, whose points are sup-lattice morphisms from M to Ω, is quite different from dual $\hat{X}$, defined for continuous lattices in general and algebraic lattices in particular, whose elements are the Scott open filters of X—see, e.g., [6].

Let M_0 be the join subsemilattice of compact elements of M. Suplattice homomorphisms from M to a frame are equivalent to join semilattice homomorphisms from M_0 to the frame, and it follows that

$$\begin{aligned}\Omega\hat{M} &= Fr\langle M_0(\text{qua} \vee\text{-}\mathbf{sL})\rangle \\ &\cong Fr\langle DL\langle M_0(\text{qua} \vee\text{-}\mathbf{sL})\rangle(\text{qua } \mathbf{DL})\rangle \\ &\cong \mathrm{Idl}(DL\langle M_0(\text{qua} \vee\text{-}\mathbf{sL})\rangle).\end{aligned}$$

The free meet semilattice over M_0 qua poset is $\mathscr{F}M_0/\sqsubseteq_U$, where $\mathscr{F}M_0$ is the finite powerset (the free semilattice with $\cup$ the binary operation) and the "upper order" $\sqsubseteq_U$ is defined by $S \sqsubseteq_U T$ if for every $y \in T$ there is some $x \in S$ with $x \leqslant y$. It is easily checked that $\mathscr{F}M_0/\sqsubseteq_U$ is not only a meet semilattice, but also a distributive lattice with

$$S \vee T = \{s \vee t \mid s \in S, t \in T\}.$$

In fact, it is isomorphic to $DL\langle M_0(\text{qua} \vee\text{-}\mathbf{sL})\rangle$. We thus have a concrete representation of $\Omega\hat{M}$ as $\mathrm{Idl}(\mathscr{F}M_0/\sqsubseteq_U)$. The injection of generators maps $x \in M$ to $\Diamond x = \bigcup\{\downarrow\{x'\} \mid x' \leqslant x, x' \in M_0\}$. If $\Diamond x \leqslant \Diamond y$, then for every compact (i.e., in M_0) $x' \leqslant x$ we have $\{x'\} \sqsubseteq_U \{y'\}$ for some compact $y' \leqslant y$, so $x' \leqslant y' \leqslant y$ and it follows that $x \leqslant y$. $\square$

4. Localic tropological systems

In the present paper we provide a more constructive approach to the issues addressed in [2,11], namely requiring P to be a locale instead of a discrete set. This approach provides us, as was already mentioned in the introduction and will become clear below, with algebraic tools otherwise unavailable, and opens the way to constructive reasoning.

4.1. Pre-systems

Definition 4.1. A *localic pre-tropological system* (P, Q, L, K), or (*localic*) *pre-system*, consists of

- a locale P, whose points are called *states*,
- a unital quantale Q, of *finite observations*,
- a unital left Q-module L, of *finitely observable properties*,
- a Q-indexed family of locale maps $P \to \mathrm{P_L}P$ that, via the order isomorphism $\mathbf{Loc}(P, \mathrm{P_L}P) \cong \mathbf{LocSL}(\mathrm{P_L}P, \mathrm{P_L}P)$, give a localic right Q-module structure on $\mathrm{P_L}P$, referred to as the *dynamics* of the system;
- a locale map $K : P \to \hat{L}$ whose extension to a homomorphism of localic sup-lattices $\mathrm{P_L}P \to \hat{L}$ is a homomorphism of localic right Q-modules, referred to as the *behaviour map*.

A pre-system (P,Q,L,K) is referred to as a *pre-system over* (Q,L), or a *pre*-(Q,L)-*system*, and the pre-system itself is often denoted only by (P,K). The pre-system is *discrete* if P is a discrete locale.

Similarly to classical tropological systems, localic pre-systems can be presented in more than one way. In particular, the map K is uniquely determined by a homomorphism $\Pi : L \to \Omega P$ of left Q-modules, which we refer to as the *property interpretation map*:

Proposition 4.2. *A pre-*(Q,L)*-system* (P,K) *can be equivalently presented as being a pair* (P,Π), *where* P *is a locale whose frame of opens* ΩP *is a left* Q*-module, and* $\Pi : L \to \Omega P$ *is a left* Q*-module homomorphism.*

Proof. Because Kleisli morphisms for $\mathrm{P_L}$ are dual to sup-lattice homomorphisms between frames, we see that the dynamics can equivalently be characterized as a left Q-module structure on ΩP. Π is defined as $\Omega K \circ \Diamond$, where $\Diamond : L \to \Omega\hat{L}$ is the injection of generators in the frame presentation of $\Omega\hat{L}$. Let $\bar{K} : \mathrm{P_L}P \to \hat{L}$ be the homomorphic extension of $K : P \to \hat{L}$. By Proposition 3.6, $\bar{K} = \hat{\Pi}$. It is a right module homomorphism if and only if for every $a \in Q$ we have $\bar{K}; a = a; \bar{K} : \mathrm{P_L}P \to \hat{L}$. Now,

$$\Omega\bar{K} \circ \Omega a(\Diamond x) = \Omega\bar{K}(\Diamond(a \cdot x)) = \Diamond\Pi(a \cdot x),$$
$$\Omega a \circ \Omega\bar{K}(\Diamond x) = \Omega a(\Diamond\Pi(x)) = \Diamond(a \cdot \Pi(x)).$$

Using the fact (Proposition 3.7, part 1) that $\Diamond : \Omega P \to \Omega \mathrm{P_L} P$ is 1-1, we see that $\bar{K}$ is a right localic Q-module homomorphism if and only if Π is a left Q-module homomorphism. $\square$

Henceforth we will consistently use the notation K and Π with the meanings above. Notice the role of the meet in ΩP. As argued in [15], this should represent an observational conjunction, but it should be thought of as implemented by repeated observational runs. To observe $x \wedge y$ of a process p, you must first be able to save a backup copy of p, then you observe x, then you reinstate the saved version of p, then you observe y.

This contrasts with the sup-lattice L whose elements represent single observational runs. Though L has a lattice theoretic meet, it does not in general distribute over joins and is not considered to have observational significance as a conjunction.

4.2. State axioms

Similarly to the situation with classical tropological systems, we wish to exclude the bottom point 0 from the image of K. The sublocale $\{0\}$ is the closed complement of the open sublocale $\Diamond\top_L$ (we identify the open sublocale of X corresponding to $x \in \Omega X$ with x itself), and so the analogue of excluding the bottom point is to require K to factor via the open sublocale $\Diamond\top_L$. In terms of frames, the open sublocale $\Diamond\top_L$ is defined by

$$\Omega\Diamond\top_L = Fr\langle \Omega\hat{L}(\text{qua } \mathbf{Fr}) \mid \top \leqslant \Diamond\top_L \rangle,$$

and K factors via $\Diamond\top_L$ if and only if Π factors through the injection of generators $L \to \Omega\hat{L}$ and the quotient $\Omega\hat{L} \to \Omega\Diamond\top_L$, which is equivalent to Π being strong. States such that $K(p) = 0$ are regarded as "nonexistent" (cf. Section 2—they have the same meaning as the empty set of states), which motivates the following terminology.

Definition 4.3. A pre-system (P, K) satisfies the *existence axiom* if K factors via the open sublocale $\Diamond\top_L$ or, equivalently, Π is strong.

However, we need to go a little further in order to get a suitable localic analogue of tropological systems. If a is a *subunit* of a unital quantale Q (i.e., $a \leqslant 1_Q$) then in a discrete system the dynamics gives us $p \cdot a \subseteq \{p\}$. Hence, insofar as $p \cdot a$ exists at all (i.e., insofar as a is possible for p), it is the whole of $\{p\}$. In non-discrete pre-systems this is not automatic, for p is no longer an atom of the specialization preorder of P. However, we would like to retain some "dynamic atomicity", meaning that if a state q equals $p \cdot a$ for some $a \leqslant 1_Q$ then q should coincide with p. In other words, the system should remain in the same state—"stable"—if only subunits are observed, and we need a new "stability axiom" to enforce this.

In the idea just described, that if a is a subunit then "insofar as $p \cdot a$ *exists* at all, it is the whole of $\{p\}$", the meaning of "existence" should be that $p \cdot a$ exists if it is a point of $\mathrm{P_L}P$ that satisfies the open $\Diamond\top$. In other words, if we let $e : \mathrm{P_L}P \to \$$ be the map to \$ corresponding to $\Diamond\top$ (\$ is the Sierpiński locale, whose frame is free on one generator and whose points are thus the truth values), then e assigns to the points of P the truth value $\top$, and $e(p \cdot a)$ is the truth value for "$p \cdot a$ exists".

Now we shall need a "selection map" $s : \$ \times P \to \mathrm{P_L}P$ such that $s(t, p)$ is "$\downarrow p$, insofar as t is true (and $\emptyset$ otherwise)". The motivation is that if $\theta_a : P \to \$ \times P$ is the pairing $\langle e \circ (- \cdot a), \mathrm{Id} \rangle$, then the map $s \circ \theta_a : P \to \mathrm{P_L}P$ assigns to each state p the point "$\downarrow p$, insofar as $p \cdot a$ exists"—and the stability axiom that we are looking for should therefore correspond to requiring

$$- \cdot a = s \circ \theta_a. \tag{1}$$

In general, \$ is the ideal completion of the two-element lattice $\{0, \top\}$ with the general point t being the directed join $\bigvee^{\uparrow}(\{0\} \cup \{\top \mid t\})$ (classically this is either 0 or $\top$). Since s must preserve directed joins, we deduce the general form of $s(t, p)$,

$$s(t, p) = \bigvee\nolimits^{\uparrow}(\{\emptyset\} \cup \{\downarrow q \mid q = p \text{ and } t\}),$$

and we have $s(0, p) = \emptyset$ and $s(\top, p) = \downarrow p$. [It actually comes from a very general result that maps from $\$ \times X$ to Y are equivalent to pairs (f, g) of maps from X to Y with $f \sqsubseteq g$—or more generally in the category of toposes, a 2-cell $f \Rightarrow g$. In other words, the exponential $Y^{\$}$ is also a lax kernel pair of Id_Y. In the present case, s corresponds to the pair $(\emptyset \circ !, \downarrow_P)$, with $\emptyset \circ ! : P \to 1 \to \mathrm{P_L}P$.]

Proposition 4.4. *Eq.* (1) *holds if and only if* $a \cdot x = a \cdot \top \wedge x$ *for all* $x \in \Omega P$.

Proof. The frame homomorphism for s is given by $\Diamond x \mapsto \{\top\} \times x$ ($\{\top\}$ is the generator of $\Omega\$$), and for $e \circ (- \cdot a)$ we have $\{\top\} \mapsto \Diamond\top \mapsto a \cdot \top$. Hence, the pairing θ_a gives a frame homomorphism $\{\top\}\times x\mapsto a\cdot\top\wedge x$, and $\Omega(s\circ\theta_a)$ is $\Diamond x\mapsto a\cdot\top\wedge x$. On the other hand, $\Omega(-\cdot a)$ is $\Diamond x\mapsto a\cdot x$, and thus (1) is equivalent to requiring $a\cdot x=a\cdot\top\wedge x$ for all $x\in\Omega P$. □

Definition 4.5. Let (P,K) be a pre-system.

1. (P,K) satisfies the *stability axiom* if for all subunits a the map $-\cdot a:P\to \mathrm{P_L}P$ factors as $s\circ\theta_a:P\to\$\times P\to\mathrm{P_L}P$, or, equivalently, $a\cdot x=a\cdot\top\wedge x$ for all $x\in\Omega P$.
2. (P,K) is a *localic tropological system* (or simply a *system*, when no confusion may arise) if it satisfies both the existence and the stability axioms, otherwise known as the first and second *state axioms*.

We have found two different but equivalent formulations of the stability axiom, formulated respectively in terms of localic right modules and of left modules. Incidentally, the latter, too, has a simple intuitive meaning, for it expresses the idea that if $a\leqslant 1$, in which case observing a should not change the state, a state p "belongs to" the open $a\cdot x$—telling us that a can be observed at p with the resulting state being in x—if and only if a can be observed at p and p itself belongs to x.

Proposition 4.6. *Let (P,Q,L,K) be a pre-system.*

1. *The stability axiom is equivalent to each of the following conditions*:
   (1) $a\cdot x\geqslant a\cdot\top\wedge x$ *for all* $x,y\in\Omega P$ *and all subunits* a;
   (2) $a\cdot x\wedge y=a\cdot(x\wedge y)$ *for all* $x,y\in\Omega P$ *and all subunits* a.
2. *The stability axiom implies distributivity of the left action over binary meets*: $a\cdot(x\wedge y)=a\cdot x\wedge a\cdot y$ *for all* $x,y\in\Omega P$ *and all subunits* a.
3. *The stability axiom implies idempotence of the left action*: $a\cdot a\cdot x=a\cdot x$ *for all* $x\in\Omega P$ *and all subunits* a.
4. *The stability axiom implies commutativity of the left action*: $a\cdot b\cdot x=b\cdot a\cdot x$ *for all* $x\in\Omega P$ *and all subunits* a *and* b.

Proof. 1(1) is immediate because $a\cdot x\leqslant a\cdot\top$ follows from $x\leqslant\top$, and $a\cdot x\leqslant x$ follows from $a\leqslant 1$. Condition 1(2) obviously implies the stability axiom (let $x=\top$), and it is implied by it because $(a\cdot x)\wedge y=(a\cdot\top\wedge x)\wedge y=a\cdot\top\wedge(x\wedge y)=a\cdot(x\wedge y)$. For condition 2 we have $a\cdot(x\wedge y)=a\cdot\top\wedge x\wedge y=(a\cdot\top\wedge x)\wedge(a\cdot\top\wedge y)=a\cdot x\wedge a\cdot y$. Condition 3 follows from $a\cdot(a\cdot x)=a\cdot\top\wedge a\cdot x=a\cdot x$. Finally, for condition 4 we have $a\cdot b\cdot x=a\cdot\top\wedge b\cdot x=a\cdot\top\wedge b\cdot\top\wedge x=b\cdot a\cdot x$. □

Now we show that, as expected, stability is automatic in discrete pre-systems:

Proposition 4.7. *The stability axiom holds trivially for discrete pre-systems.*

Proof. Let (P,K) be a discrete pre-(Q,L)-system. Let also p be a state, $a\leqslant 1$ in Q, and $X\subseteq P$. We will prove that $a\cdot P\cap X\subseteq a\cdot X$. Assume $p\in a\cdot P\cap X$. We have $a\cdot P=\bigcup_{q\in P}a\cdot\{q\}$, so $p\in a\cdot\{q\}$ for some q. But $a\cdot\{q\}\subseteq 1\cdot\{q\}=\{q\}$, so $p=q$ and hence $p\in a\cdot\{p\}\subseteq a\cdot X$. □

4.3. Semantic domains

Now we bring to the localic context the intuitions about capabilities and semantics that were described in Section 2 for classical tropological systems.

Let $S=(P,Q,L,K)$ be an arbitrary (localic) pre-system.

Definition 4.8. (1) Let $\varphi\in L$, and let p be a state, i.e., a point of P. We say that φ is a *capability* of p, and write $p\models\varphi$, if p is in the open $\Pi(\varphi)$—or, equivalently, $K(p)$ is in the open $\Diamond\varphi$. [1]

(2) Let p be a state. Then $K(p)$ is the *behaviour*, or *meaning*, of p.

(3) Let p and q be states. If $K(p)\sqsubseteq K(q)$ (equivalently, $p\models\varphi\Rightarrow q\models\varphi$ for all $\varphi\in L$), we write $p\lesssim q$, and we write $p\sim q$ if both $p\lesssim q$ and $q\lesssim p$, in which case the states p and q are said to be *behaviourally equivalent*. (Localically, $\lesssim$ and $\sim$ are sublocales of $P\times P$.)

We are thus seeing $\hat{L}$ as a *semantic domain* for pre-(Q,L)-systems. The "locale P modulo behavioural equivalence" is the image of P in $\hat{L}$, which we will denote by $P/{\sim}$, and whose frame of opens is spanned by the image $\Pi[L]$ in ΩP. In other words,

Definition 4.9. The locale $P/{\sim}$ is defined up to isomorphism by the epi-mono factorization:

$$
\begin{tikzcd}
\widehat{L} & \\
 & P/{\sim} \arrow[ul, hook'] \\
P \arrow[uu, "K"] \arrow[ur, two heads] &
\end{tikzcd}
$$

If the state axioms hold, not all the points of $\hat{L}$ can be behaviours of states of pre-(Q,L)-systems, which means that $\hat{L}$ is larger than necessary as a semantic domain. For instance, the existence axiom tells us that behaviours lie in the open sublocale $\Diamond\top_L$, but that is still too large if both state axioms hold. In order to find a semantic domain that is exactly as large as necessary for (Q,L)-systems we first introduce morphisms of (pre-)systems.

Definition 4.10. Let $\mathbf{P}=(P,K)$ and $\mathbf{P}'=(P',K')$ be pre-(Q,L)-systems. A *map* $f:\mathbf{P}\to\mathbf{P}'$ is a locale map $f:P\to P'$ such that $\mathrm{P_L}f$ is a homomorphism of localic right Q-modules (equivalently, Ωf is a homomorphism of left Q-modules) and $K=K'\circ f$ (equivalently, $\Pi=\Omega f\circ\Pi'$). The category of (Q,L)-systems is defined to be the full

[1] This still makes sense if $p:X\to P$ is a generalized point of P. The formula "$p\models\varphi$" then denotes a truth value at stage X, i.e., a map from X to the Sierpiński locale \$, got by composing $p;\Pi(\varphi):X\to P\to\$$. This uses the fact that opens of a locale correspond to maps from it to \$.

subcategory of the category of pre-(Q,L)-systems whose objects are the (Q,L)-systems, and we denote it by (Q,L)-**Sys**.

Next we prove that (Q,L)-**Sys** has a final object. A reader who wishes to skip the details of the construction may go directly to Definition 4.22.

Definition 4.11. We define the following categories:
- (Q,L)-**pFrm** is the dual of the category of pre-(Q,L)-systems;
- (Q,L)-**Frm** is the dual of the category of (Q,L)-systems, i.e., (Q,L)-**Frm** $= (Q,L)$-**Sys**$^{\mathrm{op}}$.

The morphisms in these categories are referred to as *homomorphisms* (of pre-systems and systems, respectively).

Now we study the initial objects in (Q,L)-**pFrm**.

Definition 4.12. Let Q be a unital quantale and L a unital left Q-module. By a (Q,L)-*semilattice* we mean a meet-semilattice S (with $\top$) equipped with
- a unary operation $x \mapsto a * x$, for each $a \in Q$,
- a nullary operation $\bar{\varphi}$, for each $\varphi \in L$,

and satisfying the following laws:

$$a * (b * x) = (a \cdot b) * x, \tag{2}$$

$$1 * x = x, \tag{3}$$

$$a * x \leqslant (a \vee b) * x, \tag{4}$$

$$a * (x \wedge y) \leqslant a * x, \tag{5}$$

$$\bar{\varphi} \leqslant \overline{\varphi \vee \psi}. \tag{6}$$

A *homomorphism* $h : S \to S'$ of (Q,L)-semilattices is a homomorphism of meet-semilattices (with $\top$) that also preserves the additional operations. The resulting category is denoted by (Q,L)-**sL**.

The category of (Q,L)-semilattices is a category of finitary algebras, and therefore has an initial object. Every pre-(Q,L)-system is a (Q,L)-semilattice in an obvious way, and every homomorphism between pre-(Q,L)-systems is a homomorphism of (Q,L)-semilattices. In other words, there is a forgetful functor from (Q,L)-**pFrm** to (Q,L)-**sL**, and in order to obtain an initial object of (Q,L)-**pFrm** we will show that the forgetful functor has a left adjoint. For that we introduce a notion of coverage for (Q,L)-semilattices (cf. Theorem 2.1). Recall (Section 2.1) that a precoverage on a poset S is a function assigning a set of subsets of $\downarrow(x)$ to each $x \in S$.

Definition 4.13. Let S be a (Q,L)-semilattice. A (Q,L)-*coverage* on S is a precoverage C on S, such that, for all $x \in S$, $\Phi \subseteq L$, and $A \subseteq Q$,

- $\{\bar{\varphi} \mid \varphi \in \Phi\} \in C(\overline{\bigvee \Phi})$,
- $\{a * x \mid a \in A\} \in C((\bigvee A) * x)$,

and whenever $U \in C(x)$ then, for all $y \in S$ and $a \in Q$,

- $\{y \wedge u \mid u \in U\} \in C(y \wedge x)$ ("meet-stability"),
- $\{a * u \mid u \in U\} \in C(a * x)$ ("action-stability").

Now we can state a result similar to Theorem 2.1:

Theorem 4.14. *Let S be a (Q,L)-semilattice and C a (Q,L)-coverage on S. Then,*

$$M = (Q,L)\text{-}pFrm\,\langle\; S \;(\text{qua } (Q,L)\text{-}\mathbf{sL}) \mid x = \bigvee U \;\; [U \in C(x)]\rangle$$

is order isomorphic to

$$N = SL\,\langle S \;(\text{qua poset}) \mid x = \bigvee U \;\; [U \in C(x)]\rangle\,.$$

Proof. The injection of generators $S \to M$ is monotone, it trivially respects the defining relations $x = \bigvee U$ of N (they are the same as those of M), and thus it extends uniquely to a sup-lattice homomorphism $f : N \to M$.

We also want to find a sup-lattice homomorphism in the opposite direction, so let us define a structure of pre-(Q,L)-system on N. First we remark that a (Q,L)-coverage on a (Q,L)-semilattice S is also a coverage in the sense of Theorem 2.1, and thus N is a frame. Let $a \in Q$, and let the injection of generators of N be $\eta : S \to N$. The map $a \odot (-) : S \to N$ defined by $a \odot x = \eta(a * x)$ respects the defining relations of N, for if $x = \bigvee U$ is a defining relation then so is $a * x = \bigvee\{a * u \mid u \in U\}$, due to the action-stability of C; furthermore, the map is monotone, i.e., it respects the "qua poset" requirement. Thus we obtain, by homomorphic extension, a sup-lattice endomorphism $a \cdot (-)$ on N for each $a \in Q$, such that

$$a \cdot \eta(x) = a \odot x = \eta(a * x). \tag{7}$$

This defines a unital left action of Q on N, whose unitality and associativity follow easily from conditions (2) and (3); and the distributivity on the left variable is a consequence of the second of the four items of Definition 4.13, due to which there is a defining relation $\bigvee\{a * x \mid a \in A\} = (\bigvee A) * x$ in the definition of N. Similarly, the first of these items adds a defining relation $\bigvee\{\bar{\varphi} \mid \varphi \in \Phi\} = \overline{\bigvee \Phi}$, which means that the inclusion of L into N preserves joins.

The injection of generators $\eta : S \to N$ preserves finite meets (cf. Theorem 2.1), and thus condition (7) shows that η is a homomorphism of (Q,L)-semilattices, i.e., it respects the "qua (Q,L)-**sL**" requirement in the presentation of M; besides, it trivially respects the defining relations of M, and thus extends uniquely to a homomorphism $g : M \to N$ (in (Q,L)-**pFrm**). Since both f and g restrict to the identity on S we have $g = f^{-1}$. $\square$

The set of precoverages on a poset S is ordered by pointwise inclusion:

$$C \leqslant C' \quad \Leftrightarrow \quad C(x) \subseteq C'(x) \quad \text{for all } x \in S.$$

The pointwise intersection of a nonempty family of precoverages $\{C_\alpha\}_\alpha$ defines a precoverage $C(x) = \bigcap_\alpha C_\alpha(x)$, and it is easily seen that if S is a (Q,L)-semilattice and each C_α is a (Q,L)-coverage then so is C. Furthermore, the greatest precoverage $C(x) = \mathscr{P}(\downarrow(x))$ is also a (Q,L)-coverage, and thus the (Q,L)-coverages on S form a complete lattice; in particular, there is a least (Q,L)-coverage.

Corollary 4.15. *Let S be a (Q,L)-semilattice, and let C be the least (Q,L)-coverage on S. Then,*

$$(Q,L)\text{-}\mathit{pFrm}\langle S \text{ (qua } (Q,L)\text{-}\mathbf{sL}) \ \rangle$$

is order isomorphic to

$$SL\,\langle S \text{ (qua poset)} \,|\, x = \bigvee U \ [U \in C(x)]\rangle\,.$$

Hence, the pre-(Q,L)-system freely generated by a (Q,L)-semilattice S can be concretely described as consisting of the lattice of C-ideals for the least (Q,L)-coverage on S.

Definition 4.16. Let S be a (Q,L)-semilattice, and let C denote the least (Q,L)-coverage on S. We refer to the C-ideals for this coverage as the (Q,L)-*ideals* of S. We write $\langle X\rangle$ for the (Q,L)-ideal *generated* by a set $X \subseteq S$, i.e., the least(Q,L)-ideal that contains X. The *principal* (Q,L)-ideals are the ones generated by singletons, written simply $\langle x\rangle$ for each $x \in S$. The (Q,L)-*completion* of S is the lattice of all its (Q,L)-ideals and it is denoted by (Q,L)-Idl(S).

Corollary 4.17. (1) *Let S be a (Q,L)-semilattice. Then (Q,L)-*Idl(S) *is a pre-(Q,L)-system. The action on principal (Q,L)-ideals satisfies $a \cdot \langle x\rangle = \langle a * x\rangle$.*

(2) *The assignment $S \mapsto (Q,L)$-*Idl(S) *extends to a functor which is left adjoint to the forgetful functor from (Q,L)-*__pFrm__ *to (Q,L)-*__sL__. *The unit of the adjunction sends each $x \in S$ to $\langle x\rangle$.*

(3) *There is an initial object in (Q,L)-*__pFrm__, *which coincides with the (Q,L)-completion of the initial (Q,L)-semilattice.*

We remark that a more explicit description of (Q,L)-ideals can be obtained as follows, where for each x in a (Q,L)-semilattice S and each m in the monoid coproduct $(Q,\cdot)\amalg(S,\wedge)$ we write $m(x)$ to denote the action on S that freely extends the monoid action of Q on S, given by $(a,x) \mapsto a * x$, and the action of S on itself; that is, such that

1. $a(x) = a * x$ for all $a \in Q$,
2. $y(x) = y \wedge x$ for all $y \in S$.

We shall refer to such an m as a Q-*meet* on S.

Proposition 4.18. *Let S be a (Q,L)-semilattice. A subset $I \subseteq S$ is a (Q,L)-ideal if and only if it is lower closed and satisfies the following conditions, for all $x \in S$, $\varphi \in L$, $\Phi \subseteq L$, $A \subseteq Q$, and all Q-meets m on S:*

$$\{m(\bar{\varphi}) \mid \varphi \in \Phi\} \subseteq I \Rightarrow m\left(\overline{\bigvee \Phi}\right) \in I;$$

$$\{m(a * x) \mid a \in A\} \subseteq I \Rightarrow m((\bigvee A) * x) \in I.$$

Proof. The least (Q,L)-coverage on S is the least precoverage C such that, for all $x \in S$, $\varphi \in L$, $\Phi \subseteq L$, $A \subseteq Q$, and all Q-meets m on S,

$$\{m(\bar{\varphi}) \mid \varphi \in \Phi\} \in C\left(m\left(\overline{\bigvee \Phi}\right)\right),$$

$$\{m(a * x) \mid a \in A\} \in C\left(m((\bigvee A) * x)\right).$$

Hence, $C(x)$ consists of all the sets of the form $\{m(\bar{\varphi}) \mid \varphi \in \Phi\}$, for each m and Φ such that $x = m(\overline{\bigvee \Phi})$, together with all the sets $\{m(a * x) \mid a \in A\}$, for each m and A such that $x = m((\bigvee A) * x)$. The result now follows from the definition of C-ideal. $\square$

For systems everything is similar. Each state axiom can be expressed entirely within the theory of (Q,L)-semilattices: the existence axiom simply says that the top of a (Q,L)-semilattice coincides with $\top_L$, and the stability axiom tells us that $a * \top \wedge x = a * x$ for all $a \leqslant 1_Q$. This leads to the following result.

Definition 4.19. Let S be a (Q,L)-semilattice. We say that S is *strict* if the following conditions hold:
- $\top_L$ is the top of S;
- $a * \top \wedge x = a * x$ for all $a \leqslant 1_Q$.

Theorem 4.20. *Let S be a strict (Q,L)-semilattice. Then (Q,L)-Idl(S) is a (Q,L)-system.*

Proof. The condition $\top_L = \top_S$ tells us that the injection of generators $\Pi : L \to (Q,L)$-Idl(S) is strong, i.e., the existence axiom holds. For the stability axiom, let $a \leqslant 1_Q$, and let X be a (Q,L)-ideal. In (Q,L)-Idl(S) the ideal X can be expressed as a join of [principal (Q,L)-ideals generated by] elements of S, say $X = \bigvee_i x_i$. Hence, we obtain

$$a \cdot \top \wedge X = a \cdot \top \wedge \left(\bigvee_i x_i\right) = \bigvee_i (a \cdot \top \wedge x_i) = \bigvee_i (a \cdot x_i) = a \cdot \left(\bigvee_i x_i\right) = a \cdot X,$$

that is, the stability axiom holds. $\square$

Finally, since strict (Q,L)-semilattices form another finitary algebraic theory we obtain:

Corollary 4.21. *There is an initial object in* (Q,L)-**Frm**, *which coincides with the* (Q,L)*-completion of the initial strict* (Q,L)*-semilattice*.

From here on we only deal with systems.

Definition 4.22. We denote the final (Q,L)-system (i.e., the final object in (Q,L)-**Sys**) by $Sys(Q,L)$. If Q and L are clear from the context, we write only Sys.

We can also see Sys as a semantic domain, in that each state p of a (Q,L)-system P is mapped to a unique state $p_{Q,L}$ of Sys by the unique map $! : P \to Sys$, and we can think of $p_{Q,L}$ as being the *abstract dynamic behaviour* of p. But many different states of Sys can be behaviourally equivalent, and thus Sys is not a *behaviourally abstract* semantic domain.

Definition 4.23. The locale $Sys/{\sim}$ is called the *locale of processes*, and its points are called *processes*.

Notice that both Sys and $Sys/{\sim}$ provide semantic domains for systems, but whereas the former assigns to each state p of a system an abstract representative $p_{Q,L}$ of its behaviour, with all the *dynamics included*, the latter assigns to p its behaviour $K(p)$, which is a representative of p modulo behavioural equivalence, albeit without any a priori dynamics defined on it; that is, $Sys/{\sim}$ is not necessarily a (Q,L)-system, or at least not in a canonical way.

The basic construction of $Sys/{\sim}$ by image factorization of the locale map $Sys \to \hat{L}$ is simple but gives us very little information about it. In particular cases we shall have to work hard to discover more.

4.4. Completeness criteria

Let us address the completeness criteria that were defined in [2], namely "first completeness", "second completeness", and "third completeness".

First completeness is not intrinsic to a pair (Q,L) but instead relates the behavioural preorders of tropological systems over (Q,L) to other preorders defined elsewhere for labelled transition systems, typically preorders associated to notions of process semantics such as trace semantics, simulation, bisimulation, etc. See also Section 5.2.

In [2] the notions of second and third completeness were also defined in terms of labelled transition systems and process semantics. However, contrary to first completeness, this is inessential because they can be given an intrinsic formulation, strictly in terms of tropological systems [11]. In this section we adapt this to localic systems, and we study constructive versions of some proof techniques for third completeness that were used in [2].

Let Q and L denote respectively a unital quantale and a unital left Q-module. The notions of second and third completeness in [2,11] are equivalent to the following.

Definition 4.24. The pair (Q,L) is said to be *second complete* if the following condition holds for all $a,b\in Q$: if for every (Q,L)-system (P,K) and for every $x\in\Omega P$ we have $a\cdot x\leqslant b\cdot x$, then $a\leqslant b$.

The pair is *third complete* if the following condition holds for all $\varphi,\psi\in L$: if for every (Q,L)-system (P,K) we have $\Pi(\varphi)\leqslant\Pi(\psi)$, then $\varphi\leqslant\psi$.

An advantage of the localic framework is that each of these conditions can be reduced to a question about a single system.

Proposition 4.25. (1) *Let Sys_1 be the (Q,L)-system for which ΩSys_1 is freely generated* (*in* (Q,L)-**Frm**) *by one open*, y. *Then* (Q,L) *is second complete if and only if the function $Q\to\Omega Sys_1$, $a\mapsto a\cdot y$, is* 1-1.

(2) (Q,L) *is third complete if and only if $\Pi: L\to\Omega Sys$ is* 1-1.

Proof. (1) By the freeness property, $a\cdot y\leqslant b\cdot y$ if and only if $a\cdot x\leqslant b\cdot x$ for every x in every ΩP.

(2) We have $\Pi(\varphi)\leqslant\Pi(\psi)$ in ΩSys if and only if we have $\Pi(\varphi)\leqslant\Pi(\psi)$ in every ΩP. $\square$

Let us now outline, based on this proposition, a constructive localic strategy to proving third completeness. It is the constructive analogue of the "every point is a join of pointlikes" technique that was used extensively in [2]. We assume henceforth that L is algebraic.

In the classical theory [2], within the sup-lattice dual L^{op} we had a *master transition system*. This was a set *Proc* of "pointlikes", typically defined such that p is pointlike if and only if $p\neq 0$ and for every "propositional" observation φ—essentially, $\varphi\leqslant 1$ in Q—such that $p\cdot\varphi\neq 0$ we have $p\cdot\varphi=p$. We can rephrase this when we reinterpret "L^{op}" as the localic sup-lattice dual, saying that if $p\models\varphi\cdot\top$ then $p\sqsubseteq p\cdot\varphi$, i.e., for every $x\in L$, we have $p\models\Diamond x\Rightarrow p\models\Diamond(\varphi\cdot x)$. In other words, p is in the sublocale presented by *pointlikeness* relations

$$\begin{aligned}\top &\leqslant \Diamond\top\\ \varphi\cdot\top\wedge\Diamond x &\leqslant \Diamond(\varphi\cdot x)\end{aligned}$$

Note that these are exactly analogous to the state conditions in the definition of topological system, but with the big difference that the x there was an arbitrary frame element and in particular could be a *meet* of generators from L. We shall take *Proc* to be a particular sublocale of $\hat{L}$, typically presented by the relations just described.

Because of the close relationship between the state conditions and the pointlikeness relations, it is typically trivial to show that the map $K: Sys\to\hat{L}$ factors via *Proc*, as a composite $\mathrm{Cap}; i: Sys\to Proc\to\hat{L}$ where i is the sublocale inclusion.

The function $\Pi: L\to\Omega Sys$ whose injectivity we are trying to prove can be decomposed as

$$L\xrightarrow{\Diamond}\Omega\hat{L}\xrightarrow{\Omega i}\Omega Proc\xrightarrow{\Omega\mathrm{Cap}}\Omega Sys$$

and hence it suffices to show, first, that $\diamondsuit; \Omega i$ is 1-1, and, second, that ΩCap is 1-1 (i.e., that Cap is a localic surjection).

To show $\diamondsuit; \Omega i$ *is 1-1:* It suffices (and in fact, by Proposition 3.7, it is necessary) to show injectivity of the composite

$$L \xrightarrow{\diamondsuit} \Omega\hat{L} \xrightarrow{\Omega i} \Omega Proc \xrightarrow{\diamondsuit} \Omega \mathrm{P_L} Proc$$

and this is equal to the composite

$$\begin{array}{l} L \xrightarrow{\diamondsuit} \Omega\hat{L} \xrightarrow{\Omega\sqcup} \Omega \mathrm{P_L}\hat{L} \xrightarrow{\Omega \mathrm{P_L} i} \Omega \mathrm{P_L} Proc \\ x \mapsto \diamondsuit x \mapsto \diamondsuit\diamondsuit x \mapsto \diamondsuit \Omega i(\diamondsuit x) \end{array}$$

Hence so long as L falls within the scope of Proposition 3.7 it suffices to show that $\mathrm{P_L} i; \sqcup$ is a localic surjection.

This is in fact a localic version of the classical lemma that every element of L^{op} is a join of pointlikes. The map $\mathrm{P_L} i; \sqcup$ is the homomorphic extension to $\mathrm{P_L} Proc$ of i and it calculates the joins in $\hat{L}$ of certain sublocales of *Proc*. Conceptually, therefore, to show it is a surjection is to show every point of $\hat{L}$ is a join of pointlikes. Our strategy is to show it is in fact a split localic surjection, in effect by a map taking each point x of $\hat{L}$ to the sublocale of *Proc* whose points are the pointlikes less than x. In [18] it is shown that if $f: X \to Y$ is a map of locales, then Ωf has a left adjoint $\exists_f$ if and only if $\mathrm{P_L} f$ has a right adjoint, written there as f^{-1}, which serves to describe the inverse images under f of points of $\mathrm{P_L} Y$. Such a map f is called *semiopen*. Our plan is to show that i is semiopen, and that

$$\hat{L} \xrightarrow{\downarrow} \mathrm{P_L}\hat{L} \xrightarrow{i^{-1}} \mathrm{P_L} Proc \xrightarrow{\mathrm{P_L} i} \mathrm{P_L}\hat{L} \xrightarrow{\sqcup} \hat{L}$$

is the identity on $\hat{L}$.

To show Cap *is a localic surjection*: Our aim here is to show that *Proc* is itself a (Q, L)-system, with $i = K_{Proc}$. Then, because *Sys* is a final system, we get a system morphism $\Phi: Proc \to Sys$. We have $\Phi; \mathrm{Cap}; i = \Phi; K_{Sys} = K_{Proc} = i$ and, because i is an inclusion, $\Phi; \mathrm{Cap} = \mathrm{Id}_{Proc}$ and so Cap is a split surjection.

To summarize,

Theorem 4.26. *Let* (Q, L) *be a quantale and left module*, *with L algebraic*. *Suppose*
1. $i: Proc \to \hat{L}$ *is a sublocale inclusion*;
2. $K: Sys \to \hat{L}$ *factors via Proc*;
3. i *is semiopen*;
4. $\downarrow; i^{-1}; \mathrm{P_L} i; \sqcup = \mathrm{Id}_{\hat{L}}$;
5. *Proc is a* (Q, L)-*system*, *with* $i = K_{Proc}$.

Then (Q, L) *is third complete*.

In practice, a central part of the third completeness proofs that follow this strategy is to obtain a sup-lattice presentation of $\Omega Proc$ in order to get the necessary maps to $\mathrm{P_L} Proc$. In Section 5.3 we illustrate this with the example of the failures semantics.

5. Process semantics

In this section we describe applications to process semantics which aim to show that localic tropological systems are suitable for the kind of applications originally proposed in [2,11]. First, in Section 5.1, we address localic notions of labelled transition system. Then in Section 5.2 we discuss *unique extension theorems*, which state that in various cases localic transition systems can be "identified" with localic tropological systems, and we discuss the significance of these results. Finally, Section 5.3 contains an example of the proof techniques for third completeness described in Section 4.4, for the particular case of failures semantics.

5.1. Localic transition systems

Let Act be a set. A labelled transition system over Act in the classical sense can be equivalently defined to be a binary operation $\langle - \rangle - : Act \times \mathscr{P}P \to \mathscr{P}P$ that preserves unions in the second variable,

$$\langle \alpha \rangle \bigcup_{i \in I} X_i = \bigcup_{i \in I} \langle \alpha \rangle X_i,$$

where $\langle \alpha \rangle X$ is the set of all the states $p \in P$ such that $p \xrightarrow{\alpha} q$ for some state $q \in X$. In generalising to a localic setting we are led naturally, by analogy with what we did for tropological systems, to replacing the set P by a locale, the powerset $\mathscr{P}P$ being replaced by the frame of opens ΩP. Hence, each action $\alpha \in Act$ can be seen as defining a sup-lattice endomorphism $\langle \alpha \rangle$ of the frame ΩP (the "inverse image map" of the action α), or, equivalently, a map of locales $P \to \mathrm{P_L} P$ (the "direct image map" of α). This leads to a basic definition of localic transition system that coincides with the S-locales of [2], and which we recall here for convenience:

Definition 5.1. An *S-locale* (over Act) is a locale X whose frame of opens ΩX is equipped with an Act-indexed family of unary join-preserving operations $\langle \alpha \rangle$ ($\alpha \in Act$). A *map* of S-locales $h : X \to Y$ is a locale map whose inverse image frame homomorphism Ωh preserves every $\langle \alpha \rangle$, i.e., such that $\Omega h(\langle \alpha \rangle_Y y) = \langle \alpha \rangle_X (\Omega h(y))$ for all $y \in \Omega Y$. We call the frame of opens of an S-locale an *S-frame*. By *S-homomorphism* we mean the inverse image homomorphism of a map of S-locales. The category of S-locales and their maps is denoted by S-**Loc**.

Hence, an S-locale is essentially a function from Act to $\mathbf{SL}(\Omega P, \Omega P)$, and, similarly to classical labelled transition systems, there are various alternative and equivalent definitions, such as the following:

- a function from Act to $\mathbf{Loc}(P, \mathrm{P_L} P)$;
- a sup-lattice homomorphism from $\mathscr{P}Act$ to $\mathbf{SL}(\Omega P, \Omega P)$;
- a sup-lattice homomorphism from $\mathscr{P}Act \otimes \Omega P \cong \Omega(Act \times P)$ to ΩP;
- a map from P to $\mathrm{P_L}(Act \times P)$;

- a $(Q_{\mathrm{T}}, L_{\mathrm{T}})$-system, where $Q_{\mathrm{T}} = Qu\langle Act\rangle = \mathscr{P}(Act^*)$ is the free unital quantale on *Act* (Act^* is the free monoid on *Act*), and L_{T} is the initial (with strong homomorphisms) module $Q_{\mathrm{T}} \cdot \top$ (cf. Section 2.1 and Proposition 5.3).

If we define an S-lattice over *Act* to be a bounded distributive lattice D equipped with an *Act*-indexed family of operations $\langle\alpha\rangle : D \to D$ ($\alpha \in Act$) that preserve finite joins, the ideal completion functor from distributive lattices to frames restricts to a functor from S-lattices to S-frames, which is right adjoint to the obvious forgetful functor from S-frames to S-lattices. It follows that the category of S-locales S-**Loc** has a final object Tr_{S}, whose frame of opens is the ideal completion of the initial S-lattice, and which is a final coalgebra for the functor $\mathrm{P_L}(Act \times -)$. The terminology used in S-locales, where "S" stands for *simulation*, is justified because for a large class of labelled transition systems (including the image-finite ones) the final semantics defined by final S-locales coincides with the process theoretic notion of simulation; that is, two states are assigned to the same point of Tr_{S} if and only if they are similar (more precisely, the simulation preorder is the inverse image of the specialisation preorder on Tr_{S}). Hence, we can think of the points of Tr_{S} as labelled transition systems modulo simulation. These can be represented by a certain class of trees whose edges are labelled by actions, thus justifying that we call Tr_{S} a *tree locale*.

We have replaced the notion of (discrete) labelled transition system by that of S-locale. However, as mentioned in Section 1.4, in the localic setting it is necessary to define various notions of localic transition system, whose states can be separated to a lesser or greater extent. Examples are the RS-locales of [2] for ready-simulation, the B-locales of [11] for bisimulation, or the locales in [1] for a much finer bisimulation semantics including divergence. We briefly recall the definitions of RS-locale and B-locale.

Definition 5.2. An RS-locale over *Act* is an S-locale whose frame of opens is equipped with an *Act*-indexed family of 0-ary operations $\tilde{\alpha}$ ($\alpha \in Act$), where each $\tilde{\alpha}$ is the complement of $\langle\alpha\rangle\top$, and is called the *refusal* of α. A *map* of RS-locales is just a map of S-locales between RS-locales (its inverse image homomorphism necessarily preserves refusals). The category of RS-locales and their maps is denoted by RS-**Loc**. A *B-locale* over *Act* is an S-locale X such that for all $x \in \Omega X$, if x has a complement x' then $\langle\alpha\rangle x$ has a complement, which we denote by $[\alpha]x'$. A *map* of B-locales is a map of S-locales between B-locales (whose inverse image homomorphism necessarily preserves all complements that exist). The category of B-locales and their maps is denoted by B-**Loc**.

Notice that an obvious finer "spectrum" of definitions could be given by varying the elements of an S-locale that we require to be complemented. For instance, a notion of "RS^2-locale" could be defined similarly to RS-locales but requiring also "depth 2 complements", i.e., unary operations $\widetilde{\alpha\beta}$ to be complements of $\langle\alpha\rangle\langle\beta\rangle\top$, etc.
The categories of RS-locales and B-locales can be studied in much the same way as described above for S-locales. In particular, there are final locales in each case, respectively Tr_{RS} and Tr_{B}, whose points can be thought of as "trees" modulo ready-simulation and bisimulation. See [2,11] for further details.

We conjecture that the unique S-locale maps $Tr_{\mathrm{B}} \to Tr_{\mathrm{RS}} \to Tr_{\mathrm{S}}$ are localic surjections.

5.2. Unique extension theorems

The applications of quantales and modules to process semantics in [2,11] were all based on the fact that for each equivalence E the corresponding quantale Q_E contains the set of actions Act as some of the generators, and thus a tropological system whose quantale is Q_E restricts to a labelled transition system over Act. Furthermore, for each of the process semantics E that were handled in those papers using quantales and modules, there is a quantale Q_E and a left Q_E-module L_E such that every labelled transition system over Act extends to a tropological system over (Q_E, L_E). First completeness is then the statement that the behavioural preorder of the tropological system coincides with the preorder associated with E.

The simplest example of this situation is trace semantics, whose quantale is the free quantale $Q_{\mathrm{T}} = Qu\langle Act\rangle$, and whose module is $L_{\mathrm{T}} = Q_{\mathrm{T}} \cdot \top$ (Q'_{T} in [2]). Hence, every labelled transition system over Act can be uniquely extended to a tropological system over $(Q_{\mathrm{T}}, L_{\mathrm{T}})$, and this also holds for localic systems:

Proposition 5.3 (Unique extension theorem for T). *Let P be an S-locale over Act. Then there is a unique homomorphism $\Pi : L_{\mathrm{T}} \to \Omega P$ that makes P a (localic) $(Q_{\mathrm{T}}, L_{\mathrm{T}})$-system such that $\alpha \cdot x = \langle\alpha\rangle x$ for all $\alpha \in Act$ and all $x \in \Omega P$.*

Proof. The action of Act on ΩP extends in a unique way to Q_{T} because the quantale is free, and the initiality of L_{T} (with strong homomorphisms—cf. Section 2.1) gives us a unique strong homomorphism of left Q_{T}-modules $\Pi : L_{\mathrm{T}} \to \Omega P$. □

Notice that the existence axiom is needed for the above uniqueness to hold, and it plays the same role that it does in classical tropological systems. The stability axiom was not needed, however, and in fact it is trivial in this example, since there are no subunits besides $\emptyset$ and the unit itself.

The above proposition provides a way of characterizing trace semantics for an arbitrary S-locale P: we can *define* its *trace preorder* to be the behavioural preorder $\lesssim$ of the unique $(Q_{\mathrm{T}}, L_{\mathrm{T}})$-system associated with the S-locale P, and $P/\sim$ is the associated quotient locale. Of course, for discrete systems this yields the usual definition of trace semantics.

The other process semantics addressed in [2,11] (except A, F and R, which were handled with quantaloids) can be approached in a similar way, because there are unique extension theorems for them, even though their quantales are not free on Act. The details of this will appear elsewhere, but we provide here a generic explanation about how the stability axiom enables one to prove such theorems.

Suppose (P, K) is a (Q, L)-system. If a is a subunit element of Q, then stability and existence imply that the action of a on ΩP is determined by Π, as follows:

$$a \cdot x = a \cdot \top_{\Omega P} \wedge x = \Pi(a \cdot \top_L) \wedge x.$$

In several process-theoretic examples, namely T, AT, FT, and RT, the quantale Q is generated by the actions $\alpha \in Act$ and subunits like $\alpha^{\surd}$ or $\alpha^{\times}$: so the Q-action is determined by the LTS structure and the values $\alpha^{\surd} \cdot \top_{\Omega P}$ and $\alpha^{\times} \cdot \top_{\Omega P}$. In each of these cases the quantale and module (Q, L) are such that

$$\alpha^{\surd} \cdot \top_L = \alpha \cdot \top_L \text{ in } L, \tag{8}$$

naturally telling us that being able to observe $\alpha^{\surd}$—i.e., being able to observe that α can be done—is the same capability as being able to observe α itself,

$$\alpha^{\times} \cdot \alpha = 0 \text{ in } Q, \tag{9}$$

i.e., observing that α cannot be done cannot be followed by an observation of α, and

$$\alpha \cdot \top_L \vee \alpha^{\times} \cdot \top_L = \top_L \text{ in } L, \tag{10}$$

meaning that either α can be observed or $\alpha^{\times}$ can.

Hence, from the existence axiom together with (8) and (10) we have

$$\alpha^{\surd} \cdot \top_{\Omega P} = \Pi(\alpha^{\surd} \cdot \top_L) = \Pi(\alpha \cdot \top_L) = \alpha \cdot \top_{\Omega P},$$

$$\alpha^{\times} \cdot \top_{\Omega P} \vee \alpha \cdot \top_{\Omega P} = \Pi(\alpha^{\times} \cdot \top_L \vee \alpha \cdot \top_L) = \Pi(\top_L) = \top_{\Omega P},$$

and, using the stability axiom and (9) we get

$$\alpha^{\times} \cdot \top_{\Omega P} \wedge \alpha \cdot \top_{\Omega P} = \alpha^{\times} \cdot \alpha \cdot \top_{\Omega P} = 0_{\Omega P}.$$

It follows that $\alpha^{\times} \cdot \top_{\Omega P}$ is uniquely determined as the Boolean complement of $\alpha \cdot \top_{\Omega P}$, if it exists (in RS-locales it is $\tilde{\alpha}$), which explains why unique extension theorems for FT and RT require RS-locales, whereas S-locales suffice for AT, which only uses $\alpha^{\surd}$. Classically, unique extension theorems for tropological systems were known [11,9], but the present paper enables us to adopt a different point of view, realizing that the reason why these results hold is the triviality of stability in discrete systems (cf. Proposition 4.7), and this understanding helps produce clearer and more explicit proofs of them.

Unique extension theorems are also useful from the point of view of the completeness criteria described in Section 4.4, because the reason why second and third completeness, as described in [2] with reference to labelled transition systems, is the same as in Definition 4.24, which makes no reference to transition systems, is precisely the fact that transition systems for the quantales and modules in [2] are "the same as" tropological systems, due to the unique extension theorems.

Still another application of unique extension theorems with regard to completeness criteria lies in the proof techniques for third completeness (see Section 4.4 and also Section 5.3), which are often based on showing that a certain locale *Proc* is a system. The unique extension theorems reduce this problem to that of showing that it is an S-locale, RS-locale, or B-locale, as appropriate.

5.3. Third completeness for F

Let us now apply the constructive localic approach to third completeness summarized in Theorem 4.26, in the particular case of the failures semantics F.

This is in some respects an atypical semantics, treated in [2] using quantaloids instead of quantales. We shall show that this can be replaced by a pair $(Q_{\mathrm{F}}, L_{\mathrm{F}})$ in which L_{F} is not a homomorphic image of Q_{F}. One feature of the atypicality is that third completeness for F is not fully captured by Definition 4.24, because the criterion of Definition 4.24 is too weak, and we strengthen it by taking completeness with respect to a restricted class of "standard" $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-systems. Nonetheless, the techniques used are illustrative of those for other semantics.

A central part of the calculation is going to be to obtain a sup-lattice presentation of $\Omega Proc$ in order to get the maps to $\mathrm{P_L} Proc$ that are necessary to make *Proc* a localic transition system. Already the calculations are non-trivial, though one should bear in mind that they are replacing the classical use of choice.

For the remainder of this section we shall fix a set *Act* of *actions*. *Act* need not be finite, but we do require it to have decidable equality. We define

$$Q_{\mathrm{F}} = Qu\langle Act\rangle = \mathscr{P}(Act^*),$$

$$\begin{aligned} L_{\mathrm{F}} = Q_{\mathrm{F}}\text{-Mod } \langle \mathscr{F}Act(\text{qua poset under } \supseteq) \mid \alpha\cdot\emptyset^{\times} \leqslant \emptyset^{\times}\ (\alpha\in Act),\\ U^{\times} \leqslant \alpha\cdot\emptyset^{\times} \vee (U\cup\{\alpha\})^{\times}\ (\alpha\in Act, U\subseteq_{\mathrm{fin}} Act)\rangle. \end{aligned}$$

Here we use $\mathscr{F}$ for the finite powerset, and write $U^{\times}$ for the generator of L_{F} corresponding to the finite subset U—it is to denote a conjunction of refusals of the actions in U.

Notice that the presentations of both Q_{F} and L_{F} are coherent (no infinite joins), and it follows that L_{F} is algebraic as required in Theorem 4.26.

We now prove, not a *unique* extension theorem, but a *canonical* extension theorem for $(Q_{\mathrm{F}}, L_{\mathrm{F}})$, which will be used later. (The difficulty is that because the refusals are not part of the quantale, the state axioms give us no control over them.)

Lemma 5.4. *Let P be an RS-locale. Then there is a least strong module homomorphism $\Pi: L_{\mathrm{F}}\to\Omega P$ making (P,Π) a $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-system.*

Proof. For existence, define

$$\Pi(U^{\times}) = \bigwedge_{\beta\in U}\tilde{\beta}.$$

It is easily checked that this respects the relations and is strong. Now suppose Π' is another such. We show by induction on U (this is the "$\mathscr{F}$-induction" of [17]) that $\Pi(U^{\times})\leqslant\Pi'(U^{\times})$. If $U=\emptyset$, this follows from strength. If $U=U_0\cup\{\alpha\}$, then

$$\begin{aligned} \Pi(U^{\times}) &= \tilde{\alpha}\wedge\Pi(U_0^{\times}) \leqslant \tilde{\alpha}\wedge\Pi'(U_0^{\times})\\ &\leqslant \tilde{\alpha}\wedge\alpha\cdot\Pi'(\emptyset^{\times})\vee\Pi'((U_0\cup\{\alpha\})^{\times})\leqslant\Pi'(U^{\times}). \qquad\square \end{aligned}$$

Definition 5.5. A $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-system (P, Π) is *standard* if

(1) P, already known to be an S-locale, is an RS-locale, and

(2) Π is as defined in Lemma 5.4.

Notice that a map of standard $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-systems is just a map of RS-locales. It follows that the full subcategory of $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-**Sys** whose objects are the standard $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-systems is isomorphic to RS-**Loc**.

Lemma 5.6.

$$\begin{aligned} L_{\mathrm{F}} \cong SL\ \langle Act^* \times \mathscr{F}Act \mid \\ & (s, (U \cup V)^{\times}) \leqslant (s, U^{\times}) \\ & (s \cdot t, \emptyset^{\times}) \leqslant (s, \emptyset^{\times}) \\ & (s, U^{\times}) \leqslant (s \cdot \alpha, \emptyset^{\times}) \vee (s, (U \cup \{\alpha\})^{\times})\rangle \end{aligned}$$

Proof. Let us write L'_{F} for the sup-lattice presented on the right-hand side. First, we show that this is a left Q_{F}-module. If $\gamma \in Act$ then it acts by

$$\gamma \cdot (s, U^{\times}) = (\gamma \cdot s, U^{\times})$$

and this clearly respects the relations, so it extends to a sup-lattice homomorphism. Since Q_{F} is free, this collection of homomorphisms extends to a Q_{F}-module action.

We can now define a module homomorphism $\theta : L_{\mathrm{F}} \to L'_{\mathrm{F}}$ by $U^{\times} \mapsto (1, U^{\times})$, with again an easy check that the relations are respected.

Inversely, we define a sup-lattice homomorphism $\varphi : L'_{\mathrm{F}} \to L_{\mathrm{F}}$ by $(s, U^{\times}) \mapsto s \cdot U^{\times}$. The only relation of L'_{F} that causes any trouble is the second, for which we must show that $s \cdot t \cdot \emptyset^{\times} \leqslant s \cdot \emptyset^{\times}$, and this requires an induction on t. Then, to show that φ is a module homomorphism it suffices to check

$$\gamma \cdot \varphi((s, U^{\times})) = \gamma \cdot s \cdot U^{\times} = \varphi(\gamma \cdot (s, U^{\times}))$$

Now to show $\theta ; \varphi = \mathrm{Id}$ we use

$$U^{\times} \mapsto (1, U^{\times}) \mapsto 1 \cdot U^{\times} = U^{\times}$$

while for $\varphi ; \theta = \mathrm{Id}$ we use

$$(s, U^{\times}) \mapsto s \cdot U^{\times} \mapsto s \cdot (1, U^{\times}) = (s, U^{\times}). \qquad \square$$

Corollary 5.7.

$$\begin{aligned} \Omega\widehat{L_{\mathrm{F}}} = Fr\ \langle Act^* \times \mathscr{F}Act \mid \\ & (s, (U \cup V)^{\times}) \leqslant (s, U^{\times}) \\ & (s \cdot t, \emptyset^{\times}) \leqslant (s, \emptyset^{\times}) \\ & (s, U^{\times}) \leqslant (s \cdot \alpha, \emptyset^{\times}) \vee (s, (U \cup \{\alpha\})^{\times})\rangle \end{aligned}$$

The pointlikeness conditions (taken essentially from [2]) are not exactly of the pattern mentioned in Section 4.4; we require $\alpha\cdot\top\wedge\alpha^{\times}\leqslant 0$. Hence,

Definition 5.8.

$$
\begin{aligned}
\Omega Proc_{\mathrm{F}} = Fr\ \langle Act^{*}\times\mathscr{F}Act\,|\ & \\
& (s,(U\cup V)^{\times})\leqslant(s,U^{\times})\\
& (s\cdot t,\emptyset^{\times})\leqslant(s,\emptyset^{\times})\\
& (s,U^{\times})\leqslant(s\cdot\alpha,\emptyset^{\times})\vee(s,(U\cup\{\alpha\})^{\times})\\
& \top\leqslant(1,\emptyset^{\times})\\
& (\alpha,\emptyset^{\times})\wedge(1,\{\alpha\}^{\times})\leqslant 0\rangle
\end{aligned}
$$

Lemma 5.9. *In $\Omega Proc_{\mathrm{F}}$ we have* $(1,(U\cup V)^{\times})=(1,U^{\times})\wedge(1,V^{\times})$.

Proof. First,

$$
\begin{aligned}
(1,U^{\times})\wedge(1,\alpha^{\times}) &\leqslant((\alpha,\emptyset^{\times})\vee(1,(U\cup\{\alpha\})^{\times}))\wedge(1,\{\alpha\}^{\times})\\
&=(1,(U\cup\{\alpha\})^{\times}).
\end{aligned}
$$

The full result can now be deduced by induction on V. $\square$

We can simplify the presentation by defining a partial order on $Act^{*}\times\mathscr{F}Act$:

Definition 5.10. If $(s,U^{\times})$ and $(t,V^{\times})$ are elements of $Act^{*}\times\mathscr{F}Act$, define $(s,U^{\times})\leqslant(t,V^{\times})$ if either $s=t$ and $U\supseteq V$, or t is a prefix of s (i.e., $s=t\cdot u$ for some u) and $V=\emptyset$.

Now we have an order on $\mathscr{F}(Act^{*}\times\mathscr{F}Act)$, namely the upper order $\sqsubseteq_U$ (cf. end of Section 3): if A and B are finite subsets of $Act^{*}\times\mathscr{F}Act$, then $A\sqsubseteq_U B$ if and only if for every $(t,V^{\times})\in B$ there is some $(s,U^{\times})\in A$ with $(s,U^{\times})\leqslant(t,V^{\times})$. We recall that the free meet semilattice over $Act^{*}\times\mathscr{F}Act$ qua poset is $\mathscr{F}(Act^{*}\times\mathscr{F}Act)/\sqsubseteq_U$, with meet calculated as $\cup$.

Lemma 5.11.

$$
\begin{aligned}
\Omega Proc_{\mathrm{F}}\cong SL\langle\mathscr{F}(Act^{*}\times\mathscr{F}Act)/\sqsubseteq_U\ (\text{qua poset})\,|\ & \\
& A\cup\{(s,U^{\times})\}\leqslant(A\cup\{(s\cdot\alpha,\emptyset^{\times})\})\vee(A\cup\{(s,(U\cup\{\alpha\})^{\times})\})\\
& A\leqslant A\cup\{(1,\emptyset^{\times})\}\\
& A\cup\{(\alpha,\emptyset^{\times}),(1,\{\alpha\}^{\times})\}\leqslant 0\rangle
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\Omega Proc_{\mathrm{F}} = Fr\ \langle Act^* \times \mathscr{F}Act(\text{qua poset})\,|\, & \\
& (s, U^{\times}) \leqslant (s\cdot\alpha, \emptyset^{\times}) \vee (s, (U\cup\{\alpha\})^{\times}) \\
& \top \leqslant (1, \emptyset^{\times}) \\
& (\alpha, \emptyset^{\times}) \wedge (1, \{\alpha\}^{\times}) \leqslant 0\rangle \\
\cong Fr\ \langle \mathscr{F}(Act^* \times \mathscr{F}Act)/\sqsubseteq_U\ (\text{qua}\ \wedge\text{-semilattice})\,|\, & \\
& A\cup\{(s, U^{\times})\} \leqslant (A\cup\{(s\cdot\alpha, \emptyset^{\times})\}) \vee (A\cup\{(s, (U\cup\{\alpha\})^{\times})\}) \\
& A \leqslant A\cup\{(1, \emptyset^{\times})\} \\
& A\cup\{(\alpha, \emptyset^{\times}), (1, \{\alpha\}^{\times})\} \leqslant 0\rangle
\end{aligned}
$$

This presentation is now in the form of a *site* (with a meet semilattice of generators and meet stability in the relations), and the result follows from Johnstone's coverage theorem in the form proved in [2]. □

Definition 5.12. Let A be a finite subset of $Act^* \times \mathscr{F}Act$.
1. The *head refusal* of A is defined as

$$\mathrm{hr}(A) = \bigcup\{U \mid (1, U^{\times}) \in A\}.$$

2. A is *inconsistent* if there is some $(\alpha\cdot s, U^{\times}) \in A$ with $\alpha \in \mathrm{hr}(A)$.

There are some constructive subtleties here connected with finiteness; see [17] for a discussion. For instance, to see that $\mathrm{hr}(A)$ is finite one should check that $\{U \mid (1, U^{\times}) \in A\}$ is finite. This follows because emptiness of the sequence s in $(s, U^{\times})$ is decidable.

Lemma 5.13. *If Act has decidable equality, then inconsistency is decidable.*

Proof. Its negation is $\forall(\alpha\cdot s, U^{\times}) \in A\,.\,\alpha \notin \mathrm{hr}(A)$, and $\alpha \notin \mathrm{hr}(A)$ is equivalent to $\forall \beta \in \mathrm{hr}(A)\,.\,\beta \neq \alpha$. □

Lemma 5.14. *Let A, B be finite subsets of $Act^* \times \mathscr{F}Act$, and suppose $A \sqsubseteq_U B$.*
1. $\mathrm{hr}(A) \supseteq \mathrm{hr}(B)$.
2. *If B is inconsistent then so is A.*

Proof. (1) Suppose $\alpha \in V$ with $(1, V^{\times}) \in B$, and suppose $(1, V^{\times}) \geqslant (s, U^{\times}) \in A$. We have $V \neq \emptyset$ (because $\alpha \in V$), so we deduce $s = 1$ and $V \subseteq U$, so $\alpha \in U \subseteq \mathrm{hr}(A)$.

(2) Suppose $(\alpha\cdot t, V^{\times}) \in B$ with $\alpha \in \mathrm{hr}(B)$, and suppose $(\alpha\cdot t, V^{\times}) \geqslant (s, U^{\times}) \in A$. Then $\alpha\cdot t$ is a prefix of s, and by part (1) $\alpha \in \mathrm{hr}(A)$, so it follows that A is inconsistent. □

Lemma 5.15. *$i : Proc_{\mathrm{F}} \to \widehat{L_{\mathrm{F}}}$ is semiopen, and $\downarrow; i^{-1}; \mathrm{P}_{\mathrm{L}} i; \sqcup = \mathrm{Id}_{\widehat{L_{\mathrm{F}}}}$.*

Proof. We define a sup-lattice homomorphism $\exists_i : \Omega Proc_{\mathrm{F}} \to \Omega\widehat{L_{\mathrm{F}}}$ by

$$\exists_i A = \begin{cases} 0 & \text{if } A \text{ is inconsistent,} \\ \bigwedge A \wedge (1, \mathrm{hr}(A)^{\times}) & \text{otherwise.} \end{cases}$$

It follows from Lemma 5.14 that the definition respects $\sqsubseteq_U$. It is fairly immediate that it respects the second and third relations, but the first is substantially harder. Given A, s, U and α, let us write

$$A_1 = A \cup \{(s, U^{\times})\},$$

$$A_2 = A \cup \{(s \cdot \alpha, \emptyset^{\times})\},$$

$$A_3 = A \cup \{(s, (U \cup \{\alpha\})^{\times})\}.$$

We then need to prove $\exists_i A_1 \leqslant \exists_i A_2 \vee \exists_i A_3$. Note that this is trivial whenever A_1 is inconsistent.

First, we consider the case $s \neq 1$. If either A_2 or A_3 is inconsistent then so is A_1. If all three are consistent then

$$\begin{aligned} \exists_i A_1 &= \exists_i A \wedge (s, U^{\times}) \\ &\leqslant \exists_i A \wedge ((s \cdot \alpha, \emptyset^{\times}) \vee (s, (U \cup \{\alpha\})^{\times})) \\ &= \exists_i A_2 \vee \exists_i A_3. \end{aligned}$$

Now we turn to the case $s = 1$. If A_2 is inconsistent then either A is, and hence also is A_1, or $\alpha \in \mathrm{hr}(A)$. In that case,

$$\begin{aligned} \exists_i A_1 &= \bigwedge A \wedge (1, (\mathrm{hr}(A) \cup U)^{\times}) \\ &= \bigwedge A \wedge (1, (\mathrm{hr}(A) \cup U \cup \{\alpha\})^{\times}) \\ &= \exists_i A_3. \end{aligned}$$

If A_3 is inconsistent then either A_1 is or we have some $(\alpha \cdot t, V^{\times}) \in A$. In that case

$$\bigwedge A \leqslant (\alpha \cdot t, V^{\times}) \leqslant (\alpha, \emptyset^{\times}),$$

so

$$\exists_i A_1 = \bigwedge A \wedge (1, (\mathrm{hr}(A) \cup U)^{\times}) \leqslant \bigwedge A \wedge (\alpha, \emptyset^{\times}) \wedge (1, \mathrm{hr}(A)^{\times}) = \exists_i A_2.$$

If $s = 1$ and all three are consistent, then

$$\begin{aligned} \exists_i A_1 &= \bigwedge A \wedge (1, (\mathrm{hr}(A) \cup U)^{\times}) \\ &\leqslant \bigwedge A \wedge (1, \mathrm{hr}(A)^{\times}) \wedge ((\alpha, \emptyset^{\times}) \vee (1, (\mathrm{hr}(A) \cup U \cup \{\alpha\})^{\times})) \\ &= \exists_i A_2 \vee \exists_i A_3. \end{aligned}$$

To show that i is semiopen, we must check that $\exists_i$ is left adjoint to Ωi. Using Lemma 5.9, $\exists_i; \Omega i = \mathrm{Id}_{\Omega Proc_{\mathrm{F}}}$. To show $\Omega i; \exists_i \leqslant \mathrm{Id}_{\Omega \widehat{L_{\mathrm{F}}}}$, it suffices to check on finite meets of generators and follows from the fact that $\exists_i A \leqslant \bigwedge A$. Finally, for $\downarrow; i^{-1}; \mathrm{P_L} i; \sqcup = \mathrm{Id}_{\widehat{L_{\mathrm{F}}}}$ we must check (looking at functions between frames) that $\Omega i; \exists_i$ acts as the identity function on generators $(s, U^\times)$ of $\Omega \widehat{L_{\mathrm{F}}}$, and this is clear. □

Lemma 5.16. *$Proc_{\mathrm{F}}$ is a $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-system, with $i = K_{Proc_{\mathrm{F}}}$.*

Proof. For $\gamma \in Act$ and A a finite subset of $Act^* \times \mathscr{F} Act$, we write

$$\gamma \circ A = \{(\gamma \cdot s, U^\times) \mid (s, U^\times) \in A\}.$$

Then the γ-action on $\Omega Proc_{\mathrm{F}}$ is defined by

$$\gamma \cdot A = \begin{cases} 0 & \text{if } A \text{ is inconsistent,} \\ \bigwedge \gamma \circ A \wedge (\gamma, \mathrm{hr}(A)^\times) & \text{otherwise.} \end{cases}$$

Then, much as in Lemma 5.15, one can show that this respects the sup-lattice defining relations of Lemma 5.11, and thus $Proc_{\mathrm{F}}$ is an S-locale. Clearly $(1, \{\alpha\}^\times)$ is the boolean complement of $(\alpha, \emptyset^\times)$, and so $Proc_{\mathrm{F}}$ is an RS-locale and thus (by Lemma 5.4) a $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-system in a canonical way. The fact that $i = K_{Proc_{\mathrm{F}}}$, as defined in Lemma 5.4, follows from Lemma 5.9. □

We have now finished the constructive third completeness argument for the failures semantics. If (as assumed throughout this section) Act has decidable equality, then we have:

Theorem 5.17. *$(Q_{\mathrm{F}}, L_{\mathrm{F}})$ is third complete with respect to standard $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-systems.*

Proof. For any RS-locale P, the map $K: P \to \widehat{L_{\mathrm{F}}}$ induced by Lemma 5.4 factors via $Proc_{\mathrm{F}}$. This includes the case of Sys, taken here to be the final standard $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-system (which is just the tree locale Tr_{RS} because the category of standard $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-systems is isomorphic to RS-**Loc**). We can now apply the argument of Theorem 4.26 to see that the function $\Pi: L_{\mathrm{F}} \to \Omega Sys$ is 1-1. □

Of course, it follows that $(Q_{\mathrm{F}}, L_{\mathrm{F}})$ is also third complete according to Definition 4.24, i.e., with respect to all the $(Q_{\mathrm{F}}, L_{\mathrm{F}})$-systems.

6. Conclusions

The methods of [2] appeared inescapably classical, but by topologizing (localically) we have arrived at constructive techniques, valid in toposes, for expressing and proving the completeness results in a way that preserves the intuitive understanding of the classical proofs. The notions of transition systems as models, of sup-lattice duals as semantic domains, and of master transition systems as sup-lattice bases of those domains, all survive the constructivization once it is accepted that their topologies cannot be ignored.

The treatment includes a development of algebras for the lower powerlocale monad as localic analogues of sup-lattices and quantale modules.

At the same time, more recent ideas, including tropological systems and the state axioms, have been used in order to bring unity to a diverse range of apparently ad hoc features such as the "pointlikeness" conditions.

The present paper has studied in detail the third completeness of the failures semantics, and has proposed a general approach that can be expected to apply to third completeness of other semantics.

A weakness of our present 3rd completeness proof is that it appears to require the set *Act* of actions to have decidable equality. We do not know to what extent this is an essential limitation.

References

[1] S. Abramsky, A domain equation for bisimulation, Inform. Comput. 92 (1991) 161–218.

[2] S. Abramsky, S.J. Vickers, Quantales, observational logic and process semantics, Math. Structures Comput. Sci. 3 (1993) 161–227.

[3] M. Bunge, J. Funk, Constructive theory of the lower powerlocale, Math. Struct. Comput. Sci. 6 (1996) 69–83.

[4] R.J. van Glabbeek, The linear time—branching time spectrum, in: J.C.M. Baeten, J.W. Klop (Eds.), Proc. CONCUR'90, Lecture Notes in Computer Science, Vol. 458, Springer, Berlin, 1990, pp. 278–297.

[5] R.J. van Glabbeek, The linear time—branching timespectrum II; the semantics of sequential systems with silent moves, Tech. Report, Stanford Univ., 1993. Extended abstract, in: E. Best (Ed.), Proc. CONCUR'93, Lecture Notes in Computer Science, Vol. 715, Springer, Berlin, 1993, pp. 66–81.

[6] P. Johnstone, Stone Spaces, Cambridge Stud. Adv. Math., Vol. 3, Cambridge University Press, Cambridge, 1982.

[7] A. Joyal, M. Tierney, An Extension of the Galois Theory of Grothendieck, Mem. Amer. Math. Soc., Vol. 309, American Mathematical Society, Providence, RI, 1984.

[8] J. Paseka, J. Rosický, Quantales, in: B. Coecke, D. Moore, A. Wilce (Eds.), Current Research in Operational Quantum Logic: Algebras, Categories and Languages, Fund. Theories Phys., Vol. 111, Kluwer Academic Publishers, Dordrecht, 2000, pp. 245–262.

[9] P. Resende, Quantales and observational semantics, in: B. Coecke, D. Moore, A. Wilce (Eds.), Current Research in Operational Quantum Logic: Algebras, Categories and Languages, Fund. Theories Phys., Vol. 111, Kluwer Academic Publishers, Dordrecht, 2000, pp. 263–288.

[10] P. Resende, Tropological systems are points of quantales, J. Pure Appl. Algeb. 173 (2002) 87–120.

[11] P. Resende, Quantales, finite observations and strong bisimulation, Theoret. Comput. Sci. 254 (2001) 95–149.

[12] K. Rosenthal, Quantales and Their Applications, in: Pitman Res. Notes Math. Ser., Vol. 234, Longman, Harlow, 1990.

[13] K. Rosenthal, The Theory of Quantaloids, in: Pitman Res. Notes Math. Ser., Vol. 348, Longman, Harlow, 1996.

[14] G. Sambin, Intuitionistic formal spaces: a first communication, in: D. Skordev (Ed.), Mathematical Logic and its Applications, Plenum, 1987, pp. 187–204.

[15] S.J. Vickers, Topology via Logic, Cambridge Tracts Theoret. Comput. Sci., Cambridge University Press, Cambridge, 1989 (revised edition).

[16] S.J. Vickers, Constructive points of powerlocales, Math. Proc. Cambridge Philos. Soc. 122 (1997) 207–222.
[17] S.J. Vickers, Topical categories of domains, Math. Struct. Comput. Sci. 9 (1999) 569–616.
[18] S.J. Vickers, Locales are not pointless, in: C.L. Hankin, I.C. Mackie, R. Nagarajan (Eds.), Theory and Formal Methods of Computing 1994: Proc. Second Imperial College Department of Computing Workshop on Theory and Formal Methods, Imperial College Press, London, 1994, pp. 199–216.